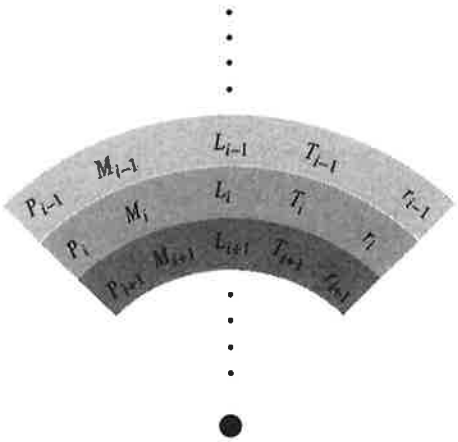

FIGURE 10.11 Zoning in a numerical stellar model. The star is assumed to be constructed of spherically symmetric mass shells, with the physical parameters associated with each zone being specified by the stellar structure equations, the constitutive relations, the boundary conditions, and the star's mass and composition. In research-quality codes some quantities are specified in the middle of mass shells (e.g., P and T), whereas others are associated with the interfaces between shells (e.g., r, M_r, and L_r).

occur close to the surface, while nuclear reactions occur near the center. By integrating in both directions, it is possible to decouple these processes somewhat, simplifying the problem.

Simultaneously matching the surface and central boundary conditions for a desired stellar model usually requires several iterations before a satisfactory solution is obtained. If the surface-to-center and center-to-surface integrations do not agree at the fitting point, the starting conditions must be changed. This is accomplished in a series of attempts, called *iterations*, where the initial conditions of the next integration are estimated from the outcome of the previous integration. A process of successive iterations is also necessary if the star is integrated from the surface to the center or from the center to the surface; in these cases the fitting points are simply the center and surface, respectively.

A very simple stellar structure code (called `StatStar`) is presented in Appendix L. `StatStar` integrates the stellar structure equations developed in this chapter in their time-independent form from the outside of the star to the center using the appropriate constitutive relations; it also assumes a constant (or homogeneous) composition throughout. Many of the sophisticated numerical techniques present in research codes have been neglected so that the basic elements of stellar model building can be more easily understood, as have the detailed calculations of the pressure equation of state and the opacity. The complex formalism of the mixing-length theory has also been left out in favor of the simplifying assumption of adiabatic convection. Despite these approximations, very reasonable models may be obtained for stars lying on the main sequence of the H–R diagram.

Polytropic Models and the Lane–Emden Equation

As we mentioned previously, it is not generally possible to solve the system of stellar structure equations and their associated constitutive relations analytically; we must employ numerical solutions to "build" stellar models. However, under very special and restrictive

situations, it is possible to find analytic solutions to a subset of the equations. The first work in this area was carried out by J. Homer Lane (1819–1880), who wrote a paper on the equilibrium of stellar configurations in the *American Journal of Science* in 1869. That work was later extended significantly by Robert Emden (1862–1940). Today, the famous equation that helps us describe analytical stellar models is referred to as the *Lane–Emden equation.*

To understand the motivation of developing the Lane–Emden equation, note that careful inspection of the stellar structure equations shows that the mechanical equations of stellar structure (Eqs. 10.6 and 10.7) could be solved simultaneously without reference to the energy equations (10.36, and either 10.68 or 10.89) if only a simple relationship existed between pressure and density. Of course, as we have seen, such a simple relationship does not generally exist; normally, temperature and composition must also enter into the pressure equation of state, often in a complicated way. However, under certain circumstances, such as for an adiabatic gas (see Eq. 10.86), the pressure can be written explicitly in terms of the density alone. Hypothetical stellar models in which the pressure depends on density in the form $P = K\rho^{\gamma}$ are known as **polytropes**. The development of polytropic models is well worth the effort since their relative simplicity allows us to gain some insight into stellar structure without all of the complications inherent in full-blown numerical models.

To derive the Lane–Emden equation, we begin with the equation for hydrostatic equilibrium, Eq. (10.6). Rewriting the equation and taking the radial derivative of both sides gives

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dM_r}{dr}.$$

We immediately see that Eq. (10.7) can be used to eliminate the mass gradient. Substituting, we get

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G(4\pi r^2\rho)$$

or

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho. \tag{10.108}$$

As an aside, it is worth pointing out here that Eq. (10.108) is actually a slightly camouflaged form of a very well-studied differential equation known as *Poisson's equation.* It is left as an exercise to show that Eq. (10.108) can be rewritten in the form

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi_g}{dr}\right) = 4\pi G\rho, \tag{10.109}$$

which is the spherically symmetric form of Poisson's equation for the gravitational potential energy per unit mass, $\Phi_g \equiv U_g/m$.[27]

[27] Poisson's equation shows up frequently in physics. For example, Gauss' Law, one of Maxwell's equations of electromagnetic theory, can be reformulated into Poisson's equation by replacing the electric field vector with the negative of the gradient of the electrostatic potential.

To solve Eq. (10.108), we now employ the relationship $P(\rho) = K\rho^{\gamma}$, where K and $\gamma > 0$ are constants. This functional form of the pressure equation is known generally as a polytropic equation of state. Substituting, taking the appropriate derivative, and simplifying, we have

$$\frac{\gamma K}{r^2}\frac{d}{dr}\left[r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] = -4\pi G\rho.$$

It is customary to rewrite the expression slightly by letting $\gamma \equiv (n+1)/n$, where n is historically referred to as the *polytropic index*. Then

$$\left(\frac{n+1}{n}\right)\frac{K}{r^2}\frac{d}{dr}\left[r^2\rho^{(1-n)/n}\frac{d\rho}{dr}\right] = -4\pi G\rho.$$

In order to simplify the last expression somewhat, it is now useful to rewrite the equation in a dimensionless form. Expressing the density in terms of a scaling factor and a dimensionless function $D(r)$, let

$$\rho(r) \equiv \rho_c[D_n(r)]^n, \qquad \text{where } 0 \le D_n \le 1.$$

(As you might suspect, ρ_c will turn out to be the central density of the polytropic stellar model.) Again substituting and simplifying, we arrive at

$$\left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{dD_n}{dr}\right] = -D_n^n.$$

Careful study of our last equation reveals that the collective constant in square brackets has the units of distance squared. Defining

$$\lambda_n \equiv \left[(n+1)\left(\frac{K\rho_c^{(1-n)/n}}{4\pi G}\right)\right]^{1/2}$$

and introducing the dimensionless independent variable ξ via

$$r \equiv \lambda_n\xi,$$

we finally arrive at

$$\boxed{\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{dD_n}{d\xi}\right] = -D_n^n,} \tag{10.110}$$

which is the famous **Lane–Emden equation**.

Solving Eq. (10.110) for the dimensionless function $D_n(\xi)$ in terms of ξ for a specific polytropic index n leads directly to the profile of density with radius $\rho_n(r)$. The polytropic

equation of state $P_n(r) = K\rho_n^{(n+1)/n}$ provides the pressure profile. In addition, if the ideal gas law and radiation pressure are assumed for constant composition (Eq. 10.20), then the temperature profile, $T(r)$, is also obtained.

In order to actually solve this second-order differential equation, it is necessary to impose two boundary conditions (which effectively specify the two constants of integration). Assuming that the "surface" of the star is that location where the pressure goes to zero (and correspondingly the density of the gas also goes to zero), then

$$D_n(\xi_1) = 0 \text{ specifies the surface at } \xi = \xi_1,$$

where ξ_1 is the location of the first zero of the solution.

Next consider the center of the star. If $r = \delta$ represents a distance infinitesimally close to the center of the star, then the mass contained within a volume of radius δ is given by

$$M_r = \frac{4\pi}{3}\overline{\rho}\,\delta^3$$

where $\overline{\rho}$ is the average density of the gas within the radius δ. Substituting into the equation for hydrostatic equilibrium, Eq. (10.6), we have

$$\frac{dP}{dr} = -G\frac{M_r\rho}{r^2} = -\frac{4\pi}{3}G\overline{\rho}^2\,\delta \to 0 \text{ as } \delta \to 0.$$

Since $P = K\rho^{(n+1)/n}$, this implies that

$$\frac{d\rho}{dr} \to 0 \text{ as } r \to 0,$$

which immediately leads to the central boundary condition

$$\frac{dD_n}{d\xi} = 0 \text{ at } \xi = 0.$$

In addition, in order for ρ_c to represent the central density of the star, it is also necessary that $D_n(0) = 1$ (this condition isn't strictly a boundary condition, it simply *normalizes* the density scaling function, D_n).

With the boundary conditions specified, it is now possible to compute the total mass of a star of a specific polytropic index. From Eq. (10.7),

$$M = 4\pi\int_0^R r^2\rho\,dr,$$

where $R = \lambda_n\xi_1$ represents the radius of the star. Rewriting in terms of the dimensionless quantities yields

$$M = 4\pi\int_0^{\xi_1}\lambda_n^2\xi^2\rho_c D_n^n\,d(\lambda_n\xi),$$

or

$$M = 4\pi\lambda_n^3\rho_c \int_0^{\xi_1} \xi^2 D_n^n \, d\xi.$$

Although this expression could be integrated directly with knowledge of $D_n(\xi)$, it can also be rewritten directly by noting, from the Lane–Emden equation and the central boundary condition, that

$$\xi^2 D_n^n = -\frac{d}{d\xi}\left[\xi^2 \frac{dD_n}{d\xi}\right]$$

gives

$$M = -4\pi\lambda_n^3\rho_c\xi_1^2 \left.\frac{dD_n}{d\xi}\right|_{\xi_1},$$

where $(dD_n/d\xi)|_{\xi_1}$ means that the derivative of D_n is evaluated at the surface.

Although the Lane–Emden equation is compact and elegant, it is important to bear in mind its many limitations. Recall that Eq. (10.110) contains no information about either energy transport or energy generation within a star; the equation only describes hydrostatic equilibrium and mass conservation, and then only within the highly idealized class of polytropic equations of state. Nevertheless, the Lane–Emden equation is capable of giving us some important insights into the structures of stars.

There are only three analytic solutions to the Lane–Emden equation, namely $n = 0$, 1, and 5. The $n = 0$ solution is given by

$$D_0(\xi) = 1 - \frac{\xi^2}{6}, \quad \text{with } \xi_1 = \sqrt{6}.$$

It is left as an exercise for you to derive the $n = 0$ solution. The solution for $n = 1$ is the well-known "sinc" function

$$D_1(\xi) = \frac{\sin\xi}{\xi}, \quad \text{with } \xi_1 = \pi,$$

and the $n = 5$ solution is given by

$$D_5(\xi) = [1 + \xi^2/3]^{-1/2}, \quad \text{with } \xi_1 \to \infty.$$

In the latter case you are asked to verify that although the radius of the star is infinite, the total mass of the star is actually finite. This is not the case for values of $n > 5$. Thus, the physical limits of n are constrained to the range $0 \leq n \leq 5$. Graphical representations of D_0, D_1, and D_5 are shown in Fig. 10.12.

This discussion of polytropes was originally motivated by the equation of state of an adiabatic gas. For the case of an ideal, monatomic gas, $\gamma = 5/3$, which implies that $n = 1.5$. In addition, as we shall see later, in Chapter 16 (see Eq. 16.12), certain extremely compressed stars in their final stage of evolution known as white dwarfs can also be described

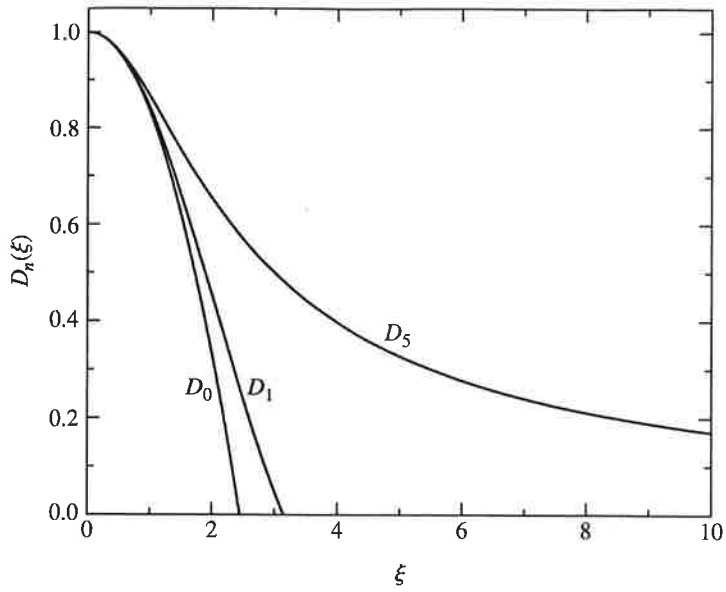

FIGURE 10.12 The analytic solutions to the Lane–Emden equation: $D_0(\xi)$, $D_1(\xi)$, and $D_5(\xi)$.

by polytropes of index 1.5 (technically these are non-relativistic, completely degenerate stars). Although the important $n = 1.5$ case cannot be solved analytically, it can be solved numerically.

Another important polytropic index is the $n = 3$ "Eddington standard model" associated with a star in radiative equilibrium. To see how this model corresponds to radiative equilibrium, consider a polytrope that is supported by both an ideal gas and radiation pressure (see Eq. 10.20). If the total pressure at a certain location in the star is represented by P, and the contribution to that pressure due to an ideal gas is given by

$$P_g = \frac{\rho kT}{\mu m_H} = \beta P, \tag{10.111}$$

where $0 \leq \beta \leq 1$, then the contribution due to radiation pressure is

$$P_r = \frac{1}{3}aT^4 = (1-\beta)P. \tag{10.112}$$

Since we are looking for a polytropic equation of state that can be expressed independent of temperature, we can combine the last two expressions to eliminate T. Solving for T in Eq. (10.111) and substituting into Eq. (10.112), we obtain

$$\frac{1}{3}a\left(\frac{\beta P \mu m_H}{\rho k}\right)^4 = (1-\beta)P.$$

This leads immediately to an expression for the total pressure in terms of the density, namely

$$P = K\rho^{4/3} \tag{10.113}$$

where

$$K \equiv \left[\frac{3(1-\beta)}{a}\right]^{1/3} \left(\frac{k}{\beta \mu m_H}\right)^{4/3}.$$

Since $\gamma = 4/3$, this implies that $n = 3$.[28]

Certainly the two most physically significant polytropic models correspond to $n = 1.5$ and $n = 3$. Although neither model can be solved analytically, the use of computers and numerical integration algorithms allow us to explore their structure and behavior relatively easily. Careful study of these polytropes can yield important insights into the structures of more realistic, although significantly more complex stellar models.

10.6 ■ THE MAIN SEQUENCE

The analysis of stellar spectra tells us that the atmospheres of the vast majority of all stars are composed primarily of hydrogen, usually about 70% by mass ($X \sim 0.7$), whereas the mass fraction of metals varies from near zero to approximately 3% ($0 < Z < 0.03$). Assuming that the initial composition of a star is **homogeneous** (meaning that the composition is the same throughout), the first set of nuclear fusion reactions ought to be those that convert hydrogen into helium (the pp chains and/or the CNO cycle). Recall that these reactions occur at the lowest temperatures because the associated Coulomb barrier is lower than that for the burning of more massive nuclei. Consequently, the structure of a homogeneous, hydrogen-rich star ought to be strongly influenced by hydrogen nuclear burning deep within its interior.

Because of the predominance of hydrogen that initially exists in the core, and since hydrogen burning is a relatively slow process, the interior composition and structure of the star will change slowly. As we saw in Example 10.3.2, a rough estimate of the hydrogen-burning lifetime of the Sun is 10 billion years. Of course, the surface conditions will not be completely static. By the Vogt–Russell theorem, any change in composition or mass requires a readjustment of the effective temperature and luminosity; *the observational characteristic of the star must change as a consequence of the central nuclear reactions*. As long as changes in the core are slow, so are the evolutionary changes in the observed surface features.[29]

Since most stars have similar compositions, the structures of stars ought to vary smoothly with mass. Recall from Examples 10.1.1 and 10.2.1 that as the mass increases, the central pressure and the central temperature should increase. Therefore, for stars of low mass, the pp chain will dominate since less energy is required to initiate these reactions than the reactions of the CNO cycle. For high-mass stars, the CNO cycle will likely dominate because of its very strong temperature dependence.

[28] We will learn in Chapter 16 that stars supported solely by a fully relativistic, completely degenerate gas can also be described by a polytropic index of 3; see Eq. (16.15).

[29] Some short-period surface changes can occur that are essentially decoupled from the long-term variations in the core. Stellar pulsations require specific conditions to exist, but their timescales are usually much shorter than the nuclear timescale. These oscillations will be discussed in Chapter 14.