where V is the volume of the gas, N is the number of particles, T is the temperature, and k is Boltzmann's constant.

Although this expression was first determined experimentally, it is informative to derive it from fundamental physical principles. The approach used here will also provide a general method for considering environments where the assumptions of the ideal gas law do not apply, a situation frequently encountered in astrophysical problems.

The Derivation of the Pressure Integral

Consider a cylinder of gas of length Δx and cross-sectional area A, as in Fig. 10.3. The gas contained in the cylinder is assumed to be composed of point particles, each of mass m, that interact through perfectly elastic collisions only—in other words, as an ideal gas. To determine the pressure exerted on one of the ends of the container, examine the result of an impact on the right wall by an individual particle. Since, for a perfectly elastic collision, the angle of reflection from the wall must be equal to the angle of incidence, the change in momentum of the particle is necessarily entirely in the x-direction, normal to the surface. From Newton's second law[1] ($\mathbf{f} = m\mathbf{a} = d\mathbf{p}/dt$) and third law, the *impulse* $\mathbf{f}\,\Delta t$ delivered to the wall is just the negative of the change in momentum of the particle, or

$$\mathbf{f}\,\Delta t = -\Delta\mathbf{p} = 2p_x\hat{\mathbf{i}},$$

where p_x is the component of the particle's initial momentum in the x-direction. Now the average force exerted by the particle over a period of time can be determined by evaluating the time interval between collisions with the right wall. Since the particle must traverse the length of the container twice before returning for a second reflection, the time interval between collisions with the same wall by the same particle is given by

$$\Delta t = 2\frac{\Delta x}{v_x},$$

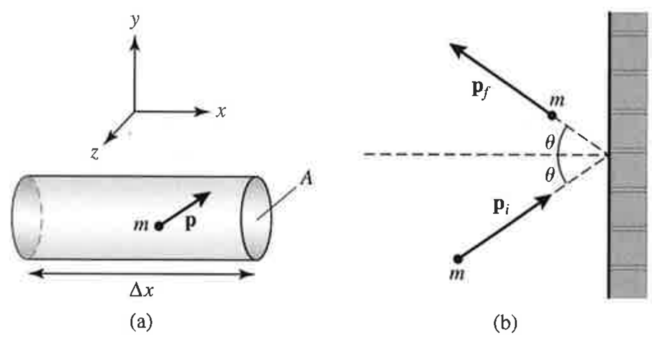

FIGURE 10.3 (a) A cylinder of gas of length Δx and cross-sectional area A. Assume that the gas contained in the cylinder is an ideal gas. (b) The collision of an individual point mass with one of the ends of the cylinder. For a perfectly elastic collision, the angle of reflection must equal the angle of incidence.

[1] Note that a lowercase **f** is used here to indicate that the force is due to a *single* particle.

so that the average force exerted on the wall by a single particle over that time period is given by

$$f = \frac{2p_x}{\Delta t} = \frac{p_x v_x}{\Delta x},$$

where it is assumed that the direction of the force vector is normal to the surface.

Now, because $p_x \propto v_x$, the numerator is proportional to v_x^2. To evaluate this, recall that the magnitude of the velocity vector is given by $v^2 = v_x^2 + v_y^2 + v_z^2$. For a sufficiently large collection of particles in random motion, the likelihood of motion in each of the three directions is the same, or $\overline{v_x^2} = \overline{v_y^2} = \overline{v_z^2} = v^2/3$. Substituting $\frac{1}{3}pv$ for $p_x v_x$, the average force per particle having momentum p is

$$f(p) = \frac{1}{3}\frac{pv}{\Delta x}.$$

It is usually the case that the particles have a range of momenta. If the number of particles with momenta between p and $p + dp$ is given by the expression $N_p\,dp$, then the total number of particles in the container is

$$N = \int_0^\infty N_p\,dp.$$

The contribution to the total force, $dF(p)$, by *all* particles in that momentum range is given by

$$dF(p) = f(p)N_p\,dp = \frac{1}{3}\frac{N_p}{\Delta x}pv\,dp.$$

Integrating over all possible values of the momentum, the total force exerted by particle collisions is

$$F = \frac{1}{3}\int_0^\infty \frac{N_p}{\Delta x}pv\,dp.$$

Dividing both sides of the expression by the surface area of the wall A gives the pressure on the surface as $P = F/A$. Noting that $\Delta V = A\,\Delta x$ is just the volume of the cylinder and defining $n_p\,dp$ to be the number of particles *per unit volume* having momenta between p and $p + dp$, or

$$n_p\,dp \equiv \frac{N_p}{\Delta V}\,dp,$$

we find that the pressure exerted on the wall is

$$P = \frac{1}{3}\int_0^\infty n_p pv\,dp. \tag{10.8}$$

This expression, which is sometimes called the **pressure integral**, makes it possible to compute the pressure, given some *distribution function*, $n_p\,dp$.

The Ideal Gas Law in Terms of the Mean Molecular Weight

Equation (10.8) is valid for both massive and massless particles (such as photons) traveling at any speed. For the special case of massive, nonrelativistic particles, we may use $p = mv$ to write the pressure integral as

$$P = \frac{1}{3}\int_0^\infty m n_v v^2 \, dv, \tag{10.9}$$

where $n_v \, dv = n_p \, dp$ is the number of particles per unit volume having speeds between v and $v + dv$.

The function $n_v \, dv$ is dependent on the physical nature of the system being described. In the case of an ideal gas, $n_v \, dv$ is the Maxwell–Boltzmann velocity distribution described in Chapter 8 (Eq. 8.1),

$$n_v \, dv = n \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv^2/2kT} 4\pi v^2 \, dv,$$

where $n = \int_0^\infty n_v \, dv$ is the particle number density. Substituting into the pressure integral finally gives

$$P_g = nkT \tag{10.10}$$

(the proof is left as an exercise in Problem 10.5). Since $n \equiv N/V$, Eq. (10.10) is just the familiar ideal gas law.

In astrophysical applications it is often convenient to express the ideal gas law in an alternative form. Since n is the particle number density, it is clear that it must be related to the mass density of the gas. Allowing for a variety of particles of different masses, it is then possible to express n as

$$n = \frac{\rho}{\overline{m}},$$

where $\overline{m}$ is the average mass of a gas particle. Substituting, the ideal gas law becomes

$$P_g = \frac{\rho kT}{\overline{m}}.$$

We now define a new quantity, the **mean molecular weight**, as

$$\mu \equiv \frac{\overline{m}}{m_H},$$

where $m_H = 1.673532499 \times 10^{-27}$ kg is the mass of the hydrogen atom. *The mean molecular weight is just the average mass of a free particle in the gas, in units of the mass of hydrogen.* The ideal gas law can now be written in terms of the mean molecular weight as

$$\boxed{P_g = \frac{\rho kT}{\mu m_H}.} \tag{10.11}$$