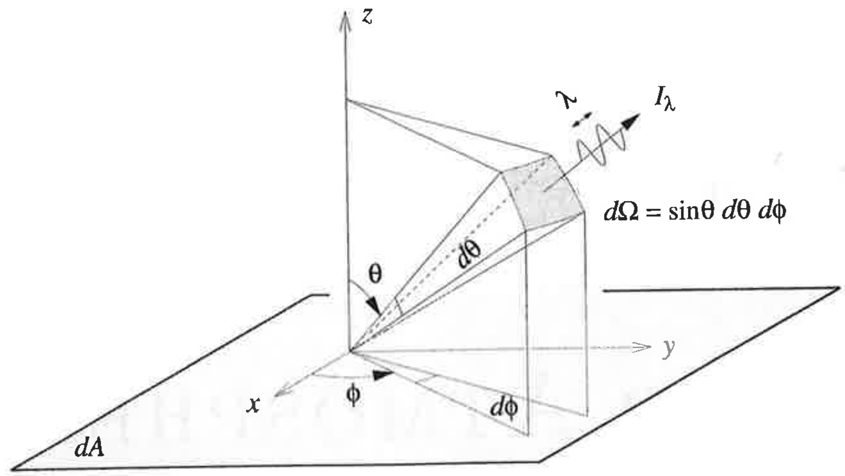

Figure 9.1 Intensity I_λ.

is the amount of electromagnetic radiation energy having a wavelength between λ and $\lambda + d\lambda$ that passes in time dt through the area dA into a solid angle $d\Omega = \sin\theta\, d\theta\, d\phi$. The specific intensity therefore has units of erg s^{-1} cm^{-3} sr^{-1}.[2] The Planck function B_λ, Eq. (3.20), is an example of the specific intensity for the special case of blackbody radiation. In general, however, the energy of the light need not vary with wavelength in the same way as it does for blackbody radiation. Later we will see under what circumstances we may set $I_\lambda = B_\lambda$.

Imagine a light ray of intensity I_λ as it propagates through a vacuum. Because I_λ is defined in the limit $d\Omega \to 0$, the energy of the ray *does not spread out (or diverge)*. The intensity is therefore *constant* along any ray traveling through empty space.

In general, the specific intensity I_λ varies with direction. The **mean intensity** of the radiation is found by integrating the specific intensity over all directions and dividing the result by 4π sr, the solid angle enclosed by a sphere, to obtain an average value of I_λ. In spherical coordinates, this average value is[3]

$$\langle I_\lambda \rangle \equiv \frac{1}{4\pi}\int I_\lambda\, d\Omega = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \sin\theta\, d\theta\, d\phi. \tag{9.1}$$

For an isotropic radiation field (one with the same intensity in all directions), $\langle I_\lambda \rangle = I_\lambda$. Blackbody radiation is isotropic and has $\langle I_\lambda \rangle = B_\lambda$.

To determine how much energy is contained within the radiation field, we can use a "trap" consisting of a small cylinder of length dL, open at both ends, with perfectly reflecting walls inside; see Fig. 9.2. Light entering the trap at

[2]Recall from Section 3.5 that erg cm^{-3} indicates an energy per unit area per unit wavelength interval, erg cm^{-2} cm^{-1}, *not* an energy per unit volume.

[3]Many texts refer to the average intensity as J_λ instead of $\langle I_\lambda \rangle$.

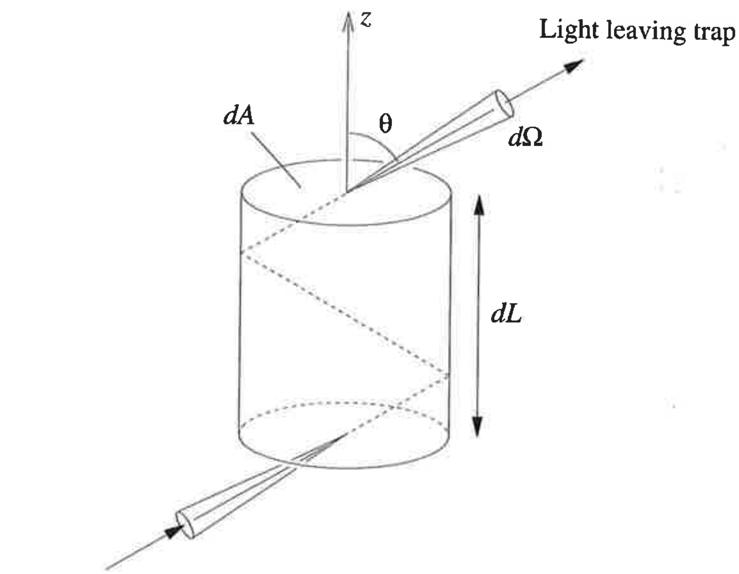

Figure 9.2 Cylindrical "trap" for measuring energy density u_λ.

one end travels and (possibly) bounces back and forth until it exits the other end of the trap. The energy inside the trap is the same as what would be present at that location if the trap were removed. The radiation that enters the trap at an angle θ travels through the trap in a time $dt = dL/(c\cos\theta)$. Thus the amount of energy inside the trap with a wavelength between λ and $\lambda + d\lambda$ that is due to the radiation that enters at angle θ is

$$E_\lambda\, d\lambda = I_\lambda\, d\lambda\, dt\, dA\, \cos\theta\, d\Omega = I_\lambda\, d\lambda\, dA\, d\Omega\, \frac{dL}{c}.$$

The quantity $dA\, dL$ is just the volume of the trap, so the specific **energy density** (energy per unit volume having a wavelength between λ and $\lambda + d\lambda$) is found by dividing $E_\lambda\, d\lambda$ by $dL\, dA$, integrating over all solid angles, and using Eq. (9.1):

$$u_\lambda\, d\lambda = \frac{1}{c}\int I_\lambda\, d\lambda\, d\Omega = \frac{1}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda\, d\lambda\, \sin\theta\, d\theta\, d\phi = \frac{4\pi}{c}\langle I_\lambda\rangle\, d\lambda. \tag{9.2}$$

For an isotropic radiation field, $u_\lambda\, d\lambda = (4\pi/c) I_\lambda\, d\lambda$, and for blackbody radiation,

$$u_\lambda\, d\lambda = \frac{4\pi}{c} B_\lambda\, d\lambda = \frac{8\pi hc/\lambda^5}{e^{hc/\lambda kT} - 1}\, d\lambda. \tag{9.3}$$

At times it may be more useful to express the energy density in terms of the frequency, ν, of the light:

$$u_\nu\, d\nu = \frac{4\pi}{c} B_\nu\, d\nu = \frac{8\pi h\nu^3/c^3}{e^{h\nu/kT} - 1}\, d\nu. \tag{9.4}$$

Thus $u_\nu\, d\nu$ is the energy per unit volume with a frequency between ν and $\nu + d\nu$.

The total energy density, u, is found by integrating over all wavelengths:

$$u = \int_0^\infty u_\lambda\, d\lambda.$$

For blackbody radiation ($I_\lambda = B_\lambda$), Eq. (3.25) shows that

$$u = \frac{4\pi}{c}\int_0^\infty B_\lambda(T)\, d\lambda = \frac{4\sigma T^4}{c} = aT^4, \tag{9.5}$$

where $a = 4\sigma/c$ is known as the *radiation constant* and has the value

$$a = 7.566 \times 10^{-15} \text{ erg cm}^{-3}\text{ K}^{-4}.$$

Another quantity of interest is F_λ, the **radiative flux**. $F_\lambda\, d\lambda$ is the *net* energy having a wavelength between λ and $\lambda + d\lambda$ that passes each second through a unit area in the direction of the z-axis:

$$F_\lambda\, d\lambda = \int I_\lambda\, d\lambda \cos\theta\, d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda\, d\lambda \cos\theta \sin\theta\, d\theta\, d\phi. \tag{9.6}$$

The factor of $\cos\theta$ determines the z-component of a light ray and allows the cancellation of oppositely directed rays. For an isotropic radiation field there is no net transport of energy, and so $F_\lambda = 0$.

Both the radiative flux and the specific intensity describe the light received from a celestial source, and the reader may wonder which of these quantities is actually measured by a telescope's photometer, pointed at the source of light. The answer depends on whether the source is resolved by the telescope. Figure 9.3(a) shows a source of light, uniform over its entire surface,[4] that is resolved by the telescope; the angle θ subtended by the source as a whole is much larger than θ_{min}, the smallest angle resolvable according to Rayleigh's criterion. In this case, what is being measured is the *specific intensity*, the amount of energy passing through the aperture area into the solid angle Ω_{min} defined by θ_{min}. For example, at a wavelength of 5010 Å, the measured value of the specific intensity at the center of the Sun's disk is

$$I_{5010} = 4.03 \times 10^{14} \text{ erg s}^{-1}\text{ cm}^{-3}\text{ sr}^{-1}.$$

Now imagine that the source is moved twice as far away. According to the inverse square law for light, Eq. (3.2), there will be only $(1/2)^2 = 1/4$ as much

[4]The assumption of a uniform light source precludes dimming effects such as limb darkening, which will be discussed later.

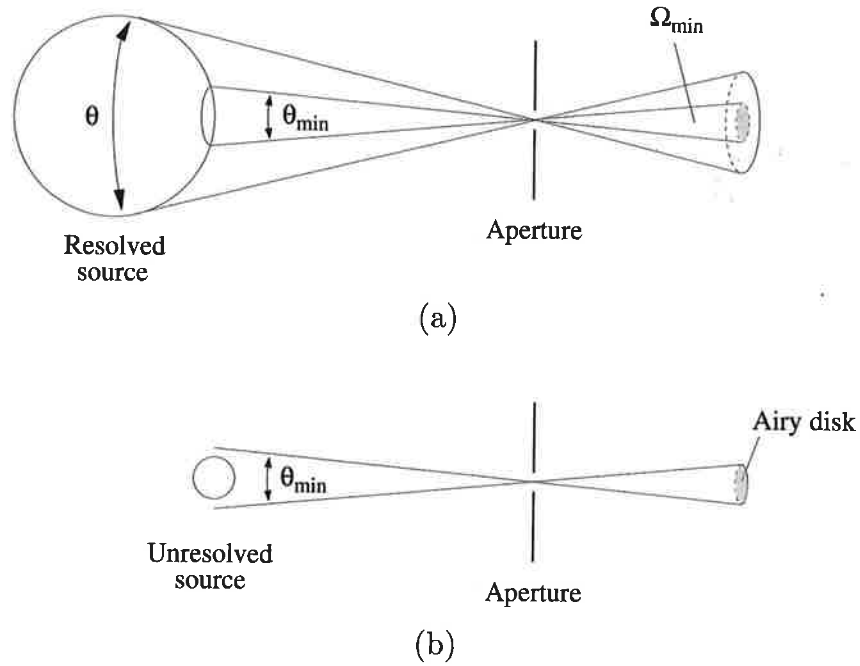

Figure 9.3 The measurement of (a) the specific intensity for a resolved source and (b) the radiative flux for an unresolved source.

energy received from each square centimeter of the source. If the source is still resolved, however, then the amount of source area that contributes energy to the solid angle Ω_{min} has increased by a factor of 4, resulting in the *same* amount of energy reaching each square centimeter of the detector. The specific intensity of light rays from the source is thus measured to be constant.[5]

However, it is the *radiative flux* that is measured for an unresolved source. As the source recedes farther and farther, it will eventually subtend an angle θ *smaller* than θ_{min}, and it can no longer be resolved by the telescope. When $\theta < \theta_{\text{min}}$, the energy received from the *entire* source will disperse throughout the diffraction pattern (the Airy disk and rings; recall Section 6.1) determined by the telescope's aperture. Because the light arriving at the detector leaves the surface of the source at all angles [see Fig. 9.3(b)], the detector is effectively integrating the specific intensity over all directions. This is just the definition of the radiative flux, Eq. (9.6). As the distance r to the source increases further, the amount of energy falling within the Airy disk (and consequently the value of the radiative flux) decreases as $1/r^2$, as expected.

A photon of energy E carries a momentum of $p = E/c$ and thus can exert a **radiation pressure**. This radiation pressure can be derived in the same

[5]This argument has been encountered previously in the statement in Section 6.1 that the image and object intensities of a resolved object are the same.

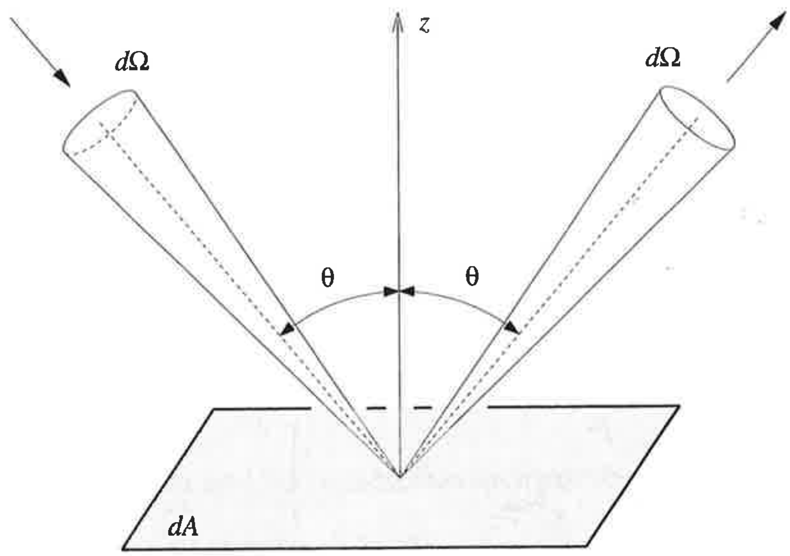

Figure 9.4 Radiation pressure produced by incident photons from the solid angle $d\Omega$.

way that gas pressure is found for molecules bouncing off a wall. Figure 9.4 shows photons reflected at an angle θ from a perfectly reflecting surface of area dA into a solid angle $d\Omega$. Because the angle of incidence equals the angle of reflection, the solid angles shown for the incident and reflected photons are the same size and inclined by the same angle θ on opposing sides of the z-axis. The change in the z-component of the momentum of photons with wavelengths between λ and $\lambda + d\lambda$ that are reflected from the area dA in a time interval dt is

$$\begin{aligned} dp_\lambda\, d\lambda &= [(p_\lambda)_{\text{final}} - (p_\lambda)_{\text{initial}}]\, d\lambda = \left[\frac{E_\lambda \cos\theta}{c} - \left(-\frac{E_\lambda \cos\theta}{c}\right)\right] d\lambda \\ &= \frac{2\,E_\lambda \cos\theta}{c}\, d\lambda = \frac{2}{c} I_\lambda\, d\lambda\, dt\, dA\, \cos^2\theta\, d\Omega. \end{aligned}$$

Dividing dp_λ by dt and dA gives $(dp_\lambda/dt)/dA$. But from Newton's second and third laws, $-dp_\lambda/dt$ is the force exerted by the photons on the area dA.[6] Thus the radiation pressure is the force per unit area, $(dp_\lambda/dt)/dA$, produced by the photons within the solid angle $d\Omega$. Integrating over the hemisphere of all incident directions results in $P_{\text{rad},\lambda}\, d\lambda$, the radiation pressure exerted by those photons having a wavelength between λ and $\lambda + d\lambda$:

$$P_{\text{rad},\lambda}\, d\lambda = \frac{2}{c}\int_{\text{hemisphere}} I_\lambda\, d\lambda\, \cos^2\theta\, d\Omega = \frac{2}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi/2} I_\lambda\, d\lambda\, \cos^2\theta\, \sin\theta\, d\theta\, d\phi.$$

[6]We will ignore the minus sign, which merely says that the force is in the $-z$-direction.

Just as the pressure of a gas exists throughout the volume of the gas and not just at the container walls, the radiation pressure of a "photon gas" exists everywhere in the radiation field. Imagine removing the reflecting surface dA in Fig. 9.4 and replacing it with a mathematical surface. The incident photons will now keep on going through dA; instead of reflected photons, photons will be streaming through dA from the other side. Thus, for an *isotropic radiation field*, there will be no change in the expression for the radiation pressure if the leading factor of 2 (which originated in the change in momentum upon reflection of the photons) is removed and the angular integration is extended over all solid angles:

$$P_{\text{rad},\lambda}\, d\lambda = \frac{1}{c}\int_{\text{sphere}} I_\lambda\, d\lambda \cos^2\theta\, d\Omega \tag{9.7}$$

$$= \frac{1}{c}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda\, d\lambda \cos^2\theta \sin\theta\, d\theta\, d\phi$$

$$= \frac{4\pi}{3c} I_\lambda\, d\lambda. \tag{9.8}$$

However, it may be that the radiation field is *not* isotropic. In that case, Eq. (9.7) for the radiation pressure is still valid but the pressure depends on the orientation of the mathematical surface dA.

The total radiation pressure produced by photons of all wavelengths is found by integrating Eq. (9.8):

$$P_{\text{rad}} = \int_0^\infty P_{\text{rad},\lambda}\, d\lambda.$$

For blackbody radiation, it is left as a problem to show that

$$P_{\text{rad}} = \frac{4\pi}{3c}\int_0^\infty B_\lambda(T)\, d\lambda = \frac{4\sigma T^4}{3c} = \frac{1}{3}aT^4 = \frac{1}{3}u. \tag{9.9}$$

Thus the **blackbody radiation pressure** is one-third of the energy density. (For comparison, the pressure of an ideal monatomic gas is two-thirds of its energy density.)

9.2 Stellar Opacity

The classification of stellar spectra is an ongoing process. Even the most basic task, such as finding the surface temperature of a particular star, is complicated by the fact that stars are not actually blackbodies. The Stefan–Boltzmann relation, in the form of Eq. (3.17), defines a star's effective temperature, but some