4. SHOCKS: STRONG NONLINEAR WAVES IN THE INTERSTELLAR MEDIUM

The problem for this section of the course is: "A supernova explodes, releasing 10^{51} erg. What is the temperature behind the shock as a function of time?" (Spitzer, pp. 246–269)

We have talked about stars and seen that massive stars end their (obvious) lives in explosions of "supernovae" that eject matter into the interstellar medium. Now, we will examine what happens to the interstellar medium when these explosions hit it. This is the problem of the motion of a disturbance in the interstellar medium: how can we solve it? We will need the equations of fluid dynamics (ignoring magnetic fields: although these fields may be important, we can obtain the essential properties of supernovae without this complication).

What is fluid dynamics?

We describe a fluid by its density, ρ, velocity, $\mathbf{u}$, and pressure, P (or another thermodynamic variable ... entropy per unit mass, temperature, ...). ρ, $\mathbf{u}$, and P are five variables, so we will need five equations. These are:

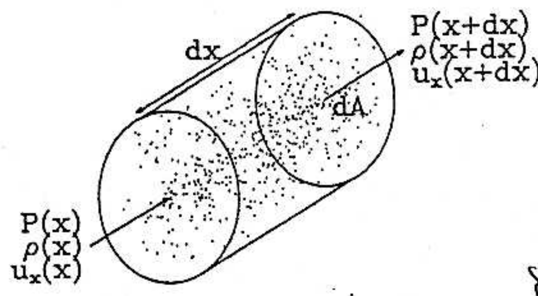
mass conservation	(equation of continuity)	scalar
momentum conservation	(Euler equation: a force/acceleration equation)	vector
energy/entropy conservation	(equation of state)	scalar

and provide $1+3+1$ equations, as required. They are the *dynamical* counterparts of the *static* equations of stellar structure.

4.1. FLUID DYNAMICAL EQUATIONS

We will derive these equations in one dimension (which is what we will be interested in most of the time) and quote results for three dimensions. We will use a simple physical method of getting the equations; *far* more sophisticated and complicated methods exist.

4.1.1. Mass Conservation



$$\text{in } dt, \text{ the mass flowing into the cylinder} = [\rho(x)\, u_x(x)]\, dA\, dt$$
$$\text{out of the cylinder} = [\rho(x+dx) u_x(x+dx)]\, dA\, dt \quad .$$

Therefore, the mass change in the cylinder in dt

$$= dA\, dt \left\{ \rho u_x\big|_x - \rho u_x\big|_{x+dx} \right\}$$
$$= -dA\, dt \, . \, \frac{\partial}{\partial x}(\rho u_x) \, . \, dx \quad .$$

But the mass in the cylinder is $dA\, dx\, \rho$. Hence, the rate of change of mass in the cylinder is

$$dA\, dx \frac{\partial \rho}{\partial t} = -dA\, dx \frac{\partial}{\partial x}(\rho u_x) \quad ,$$

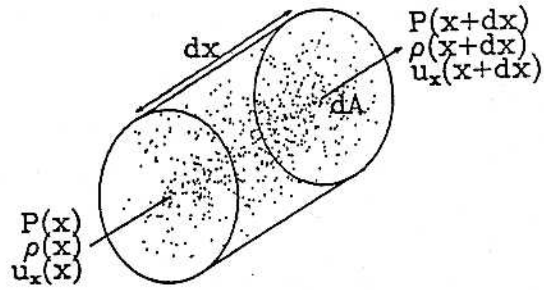
i.e.,

$$\boxed{\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0 \quad .} \qquad \text{(one-dimensional)}$$

In three dimensions, this becomes

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{u}) = 0 \quad .} \qquad \text{(three-dimensional)}$$

4.1.2. Momentum Equation



in dt, the momentum flowing into the cylinder $= \rho(x)u_x(x) \,.\, [u_x(x) \,.\, dA\, dt]$

out of the cylinder $= \rho(x+dx)\, u_x(x+dx) \,.\, [u_x(x+dx) \,.\, dA\, dt]$.

Therefore, the net momentum change in the cylinder

$$= \left[\rho u_x^2 \big|_x - \rho u_x^2 \big|_{x+dx} \right] dA\, dt$$

$$= -dA\, dt \frac{\partial}{\partial x} (\rho u_x^2)\, dx \quad .$$

The change of momentum in the cylinder

$$= \frac{\partial}{\partial t} (\rho u_x\, dA\, dx)\, dt = \frac{\partial}{\partial t} (\rho u_x) dA\, dx\, dt$$

The net pressure force on the cylinder

$$= P(x)\, dA - P(x+dx)\, dA = -\frac{\partial P}{\partial x}\, dA\, dx$$

(rate of change of momentum in cylinder) = (force) + (rate of change of momentum caused by momentum flux), i.e.,

$$\frac{\partial}{\partial t}(\rho u_x) = -\frac{\partial}{\partial x}(\rho u_x^2) - \frac{\partial P}{\partial x} \quad ,$$

i.e.,

$$u_x \frac{\partial \rho}{\partial t} + \rho \frac{\partial u_x}{\partial t} = -u_x \frac{\partial (\rho u_x)}{\partial x} - \rho u_x \frac{\partial u_x}{\partial x} - \frac{\partial P}{\partial x} \quad ,$$

but

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial x} (\rho u_x) \quad .$$

Therefore,

$$\rho \frac{\partial u_x}{\partial t} = -\rho u_x \frac{\partial u_x}{\partial x} - \frac{\partial P}{\partial x} \quad ,$$

or

$$\rho\left(\frac{\partial u_x}{\partial t}+u_x\frac{\partial u_x}{\partial x}\right)=-\frac{\partial P}{\partial x} \quad . \qquad \text{(one-dimensional)}$$

In three dimensions, this is

$$\rho\left[\frac{\partial \mathbf{u}}{\partial t}+(\mathbf{u}\,.\,\nabla)\mathbf{u}\right]=-\nabla P \quad . \qquad \text{(three-dimensional)}$$

Where the vector operation

$$(\mathbf{u}\,.\,\nabla)\mathbf{v}\equiv\left(u_x\frac{\partial v_x}{\partial x}+u_y\frac{\partial v_x}{\partial y}+u_z\frac{\partial v_x}{\partial z},\, u_x\frac{\partial v_y}{\partial x}+u_y\frac{\partial v_y}{\partial y}+u_z\frac{\partial v_y}{\partial z},\, u_x\frac{\partial v_z}{\partial x}+u_y\frac{\partial v_z}{\partial y}+u_z\frac{\partial v_z}{\partial z}\right) \quad .$$

We often write

$$\frac{D}{Dt}\equiv\left(\frac{\partial}{\partial t}+\mathbf{u}\,.\,\nabla\right) \quad ,$$

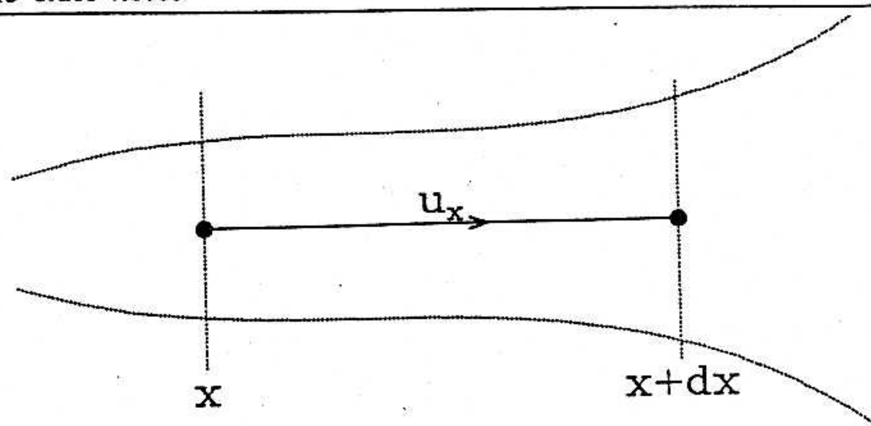
so that these equations would become $\frac{\partial}{\partial t}+v\frac{\partial}{\partial x}$

$$\rho\frac{Du_x}{Dt}=-\frac{\partial P}{\partial x} \qquad \text{or} \qquad \rho\frac{D\mathbf{u}}{Dt}=-\nabla P \quad .$$

What is the difference between $\frac{D}{Dt}$ and $\frac{\partial}{\partial t}$?

$\frac{D}{Dt}$ = <u>convective derivative</u> – rate of change of something as seen by an observer moving with the fluid.

$\frac{\partial}{\partial t}$ = <u>partial derivative</u> – rate of change of something as seen by an observer at rest watching the fluid flow past.



Consider a quantity Q. An observer at x sees Q in a fluid opposite him change at a rate $\frac{\partial Q}{\partial t}$. An observer moving with the fluid, i.e., at speed u_x, sees a change in Q due to a change with *time* and due to the fact that at dt later, he is looking at the fluid at a *different point*: $x + dx$, where $dx = u_x dt$. In other words, the change seen by moving observer is

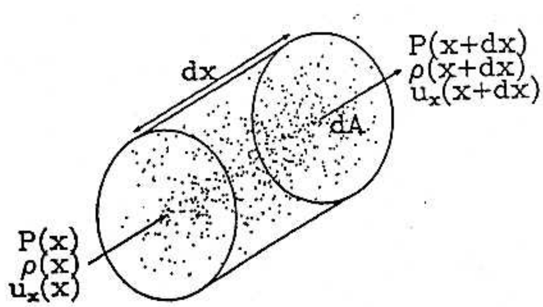
$$dt\,\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\,dx = \left(\frac{\partial Q}{\partial t} + \frac{\partial Q}{\partial x}\,u_x\right)\,dt \quad ,$$

i.e.,

$$\frac{DQ}{Dt} \equiv \frac{\partial Q}{\partial t} + u_x\,\frac{\partial Q}{\partial x} \quad .$$

The extension to three dimensions is obvious.

4.1.3. Energy/Entropy Equation



Define e = internal energy/unit mass, and ϵ = energy input/unit mass/unit time. Then, in dt, the energy flowing into the cylinder $= u_x \rho \left(e + \frac{1}{2} u_x^2 \right) \Big|_x \, dA \, dt$

out of the cylinder $= u_x \rho \left(e + \frac{1}{2} u_x^2 \right) \Big|_{x+dx} \, dA \, dt$

Thus,

work done <u>on</u> gas in cylinder = (force on cylinder)×
(distance moved by point of application of force)

$$= P u_x \big|_x \, dt \, dA$$

work done <u>by</u> gas in cylinder $= P u_x \big|_{x+dx} \, dt \, dA$.

A final contribution to the changing energy content of the cylinder is the energy generated inside the volume, $\rho \epsilon \, . \, dA \, dx \, . \, dt$.

So, the <u>total</u> energy change in the cylinder is

$$-\frac{\partial}{\partial x} \left[\rho u_x \left(e + \frac{1}{2} u_x^2 \right) \right] dA \, dx \, dt - \frac{\partial}{\partial x} (P u_x) \, dA \, dx \, dt + \rho \epsilon \, . \, dA \, dx \, dt \quad ,$$

and this must equal

$$\frac{\partial}{\partial t} \left[\rho \left(e + \frac{1}{2} u_x^2 \right) \right] dA \, dx \, dt \quad .$$

Hence,

$$\boxed{\frac{\partial}{\partial t} \left[\rho \left(e + \frac{1}{2} u_x^2 \right) \right] = -\frac{\partial}{\partial x} \left[\rho u_x \left(w + \frac{1}{2} u_x^2 \right) \right] + \rho \epsilon \quad .} \qquad \text{(one-dimensional)}$$

$$\boxed{\frac{\partial}{\partial t} \left[\rho \left(e + \frac{1}{2} u^2 \right) \right] + \nabla . \left[\rho \mathbf{u} \left(w + \frac{1}{2} u^2 \right) \right] = \rho \epsilon \quad .} \qquad \text{(three-dimensional)}$$

Where I have introduced w, the enthalpy per unit mass: $w = e + \frac{P}{\rho}$.

It is convenient to simplify this! Work with the one-dimensional equation. Expanding, it becomes

$$\rho\frac{\partial}{\partial t}\left(e+\frac{1}{2}u_x^2\right)+\left(e+\frac{1}{2}u_x^2\right)\frac{\partial\rho}{\partial t}+\rho u_x\frac{\partial}{\partial x}\left(e+\frac{1}{2}u_x^2\right)+\left(e+\frac{1}{2}u_x^2\right)\frac{\partial}{\partial x}(\rho u_x)$$
$$+P\frac{\partial u_x}{\partial x}+u_x\frac{\partial P}{\partial x}=\rho\epsilon \quad ,$$

i.e.,

$$\rho\frac{\partial e}{\partial t}+\rho u_x\frac{\partial u_x}{\partial t}+\rho u_x\frac{\partial e}{\partial x}+\rho u_x^2\frac{\partial u_x}{\partial x}+P\frac{\partial u_x}{\partial x}+u_x\frac{\partial P}{\partial x}=\rho\epsilon \quad ,$$

i.e.,

$$\rho\left(\frac{\partial e}{\partial t}+u_x\frac{\partial e}{\partial x}\right)+P\frac{\partial u_x}{\partial x}=\rho\epsilon \quad .$$

But the mass conservation equation

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho u_x)=0$$

implies that

$$\frac{\partial u_x}{\partial x}=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial t}+u_x\frac{\partial\rho}{\partial x}\right)=-\frac{1}{\rho}\frac{D\rho}{Dt} \quad .$$

Therefore, eliminating $\frac{\partial u_x}{\partial x}$,

$$\rho\frac{De}{Dt}-\frac{P}{\rho}\frac{D\rho}{Dt}=\rho\epsilon$$

or

$$\boxed{\frac{De}{Dt}-\frac{P}{\rho^2}\frac{D\rho}{Dt}=\epsilon \quad ,}$$

for the one- or three-dimensional case. We can simplify this further. Thermodynamics tells us that

$$\begin{aligned} de &= Tds - Pdv \\ &= Tds + \frac{P}{\rho^2}d\rho \quad . \end{aligned}$$

s = entropy

Hence,

$$Tds = de - \frac{P}{\rho^2}d\rho \quad ,$$

and

$$T\frac{Ds}{Dt} = \epsilon \ ,$$

entropy formulation

also for the one- or three-dimensional case. "Entropy arises due to the injection of energy" ... this is simply the statement $ds = \dfrac{\delta Q}{T}$!

Therefore, the three equations of fluid mechanics are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla . (\rho \mathbf{u}) = 0$$

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\right) = -\frac{\partial P}{\partial x} \qquad \rho\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} . \nabla)\mathbf{u}\right] = -\nabla P$$

$$T\left(\frac{\partial s}{\partial t} + u_x\frac{\partial s}{\partial x}\right) = \epsilon \qquad T\left[\frac{\partial s}{\partial t} + (\mathbf{u} . \nabla)s\right] = \epsilon$$

(one-dimensional) (three-dimensional)

Note that these equations are nonlinear, even in one dimension. This is responsible for the wide variety of possible fluid motions, and some of the charm and difficulty of the subject.

Often we do not use or acknowledge the energy/entropy equation at all; if there is no internal energy generation (so that there is no dissipation, and generation, of heat and $\epsilon = 0$), then the entropy equation can often be replaced by an *equation of state* $P(\rho)$.

Note that the $P(\rho)$ equation is in addition to $P = \dfrac{\rho k T}{\mu m_H}$ or whatever; an example might be $P \propto \rho^\gamma$. This is really a statement of entropy conservation:

$$s \propto \ln(P\rho^{-\gamma}) \ ,$$

so if $P \propto \rho^\gamma$, $\dfrac{Ds}{Dt} = 0$. In cases where there is no dissipation or energy generation, we will therefore use the adiabatic equation of state for the fluid. (But beware: this is not generally possible. It only applies to flows where $P \propto \rho^\gamma$ and the constant of proportionality is the same everywhere in the fluid: isentropic flows.)

4.2. SOUND WAVES

The simplest solution of any set of physical equations is generally that for small disturbances from equilibrium. This is usually a *wave motion* (or SHM) solution. In a fluid, we call these disturbances *sound waves.* Sound waves are the way that different parts of a fluid get to know about one another (they exchange information – e.g., about pressure – at a speed equal to the speed of sound).

We obtain sound waves in their simplest form by assuming an equation of state, $P(\rho)$, and considering small perturbations of the fluid properties about their equilibrium values, P_0 = constant, ρ_0 = constant, $u_0 = 0$.

$$\begin{aligned} \rho &= \rho_0 + \rho_1(x,t) \\ P &= P_0 + P_1(x,t) \\ u_x &= \quad\;\; u_1(x,t) \end{aligned}$$

$$\uparrow$$

small perturbations : $\rho_1 \ll \rho_0 \quad , \quad P_1 \ll P_0$

Since the fluid is initially at rest, $u_0 = 0$. We now substitute these forms for ρ, P, and u_x into the equations of fluid motion, and drop all terms that are the product of two small quantities.

$$\frac{\partial}{\partial t}(\rho_0 + \rho_1) + \frac{\partial}{\partial x}[(\rho_0 + \rho_1)u_1] = 0 \quad . \qquad \text{mass conservation}$$

Therefore,

$$\frac{\partial \rho_1}{\partial t} + \rho_0 \frac{\partial u_1}{\partial x} = 0 \quad .$$

$$(\rho_0 + \rho_1)\left[\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x}\right] = -\frac{\partial}{\partial x}(P_0 + P_1) \quad . \qquad \text{Euler's equation}$$

Therefore,

$$\rho_0 \frac{\partial u_1}{\partial t} = -\frac{\partial P_1}{\partial x} \quad .$$

We also need a $\frac{P}{\rho}$ relation:

$$P = P(\rho) \quad , \qquad \text{equation of state}$$

from which

$$\begin{aligned} P_0 + P_1 &= P(\rho_0 + \rho_1) \\ &= P(\rho_0) + \rho_1 \left(\frac{\partial P}{\partial \rho}\right)_0 \\ &= P_0 + \rho_1 \left(\frac{\partial P}{\partial \rho}\right)_0 \ . \end{aligned}$$

Thus,

$$P_1 = \rho_1 \left(\frac{\partial P}{\partial \rho}\right)_0 \ .$$

Using these equations, which are *homogeneous* in first order quantities, we first eliminate u_1 to obtain

$$\frac{\partial^2 \rho_1}{\partial t^2} = \frac{\partial^2 P_1}{\partial x^2} \ ,$$

and then eliminate P_1, to get

$$\boxed{\frac{\partial^2 \rho_1}{\partial t^2} = \left(\frac{\partial P}{\partial \rho}\right)_0 \frac{\partial^2 \rho_1}{\partial x^2} \ .}$$

This is a one-dimensional *wave equation*; if we look for wavelike solutions, $\rho_1 \propto e^{i(kx-\omega t)}$, we find that they exist if the angular frequency, ω, and wavenumber, k, are related by

$$\omega^2 = k^2 \left(\frac{\partial P}{\partial \rho}\right)_0 \ .$$

But the wave speed (the speed of sound in this case) is $c_s = \frac{\omega}{k}$; so the speed of a sound wave is given by

$$c_s^2 = \left(\frac{\partial P}{\partial \rho}\right)_0 \ .$$

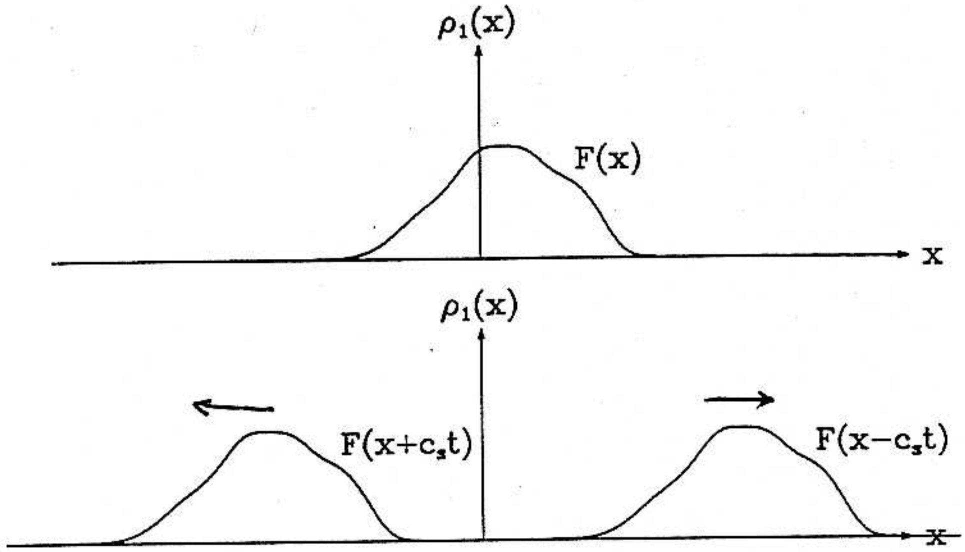
In fact, the general solution of

$$\frac{\partial^2 \rho_1}{\partial t^2} = c_s^2 \frac{\partial^2 \rho_1}{\partial x^2}$$

is

$$\rho_1 = F(x \pm c_s t) \ ,$$

which corresponds to traveling waves moving at speed c_s in a positive x direction $(x - ct)$ or in a negative x direction $(x + ct)$. F is an arbitrary function, so that waves of arbitrary shape travel coherently, without dispersion.



If the gas is adiabatic,

$$P \propto \rho^{\gamma} \ ,$$

and so

$$\left(\frac{dP}{d\rho}\right)_0 = \frac{\gamma P_0}{\rho_0} \ .$$

Hence,

$$c_s = \left(\frac{\gamma P_0}{\rho_0}\right)^{1/2} \ ,$$

or, using the ideal gas law,

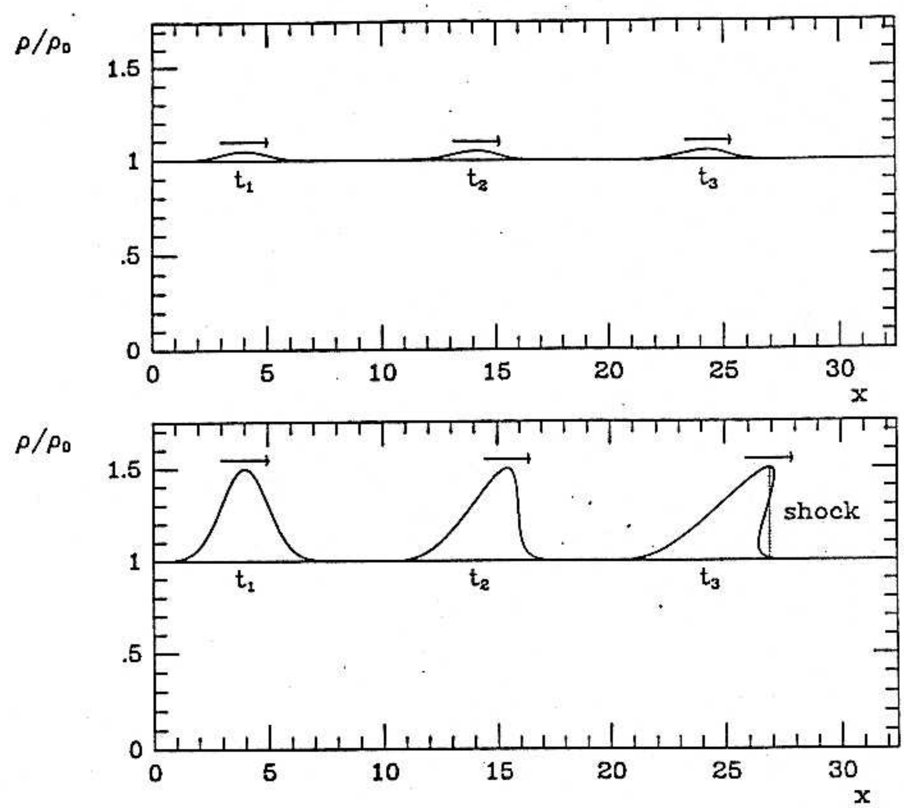
$$P_0 = \frac{\rho_0 k T_0}{m} \ ,$$

so that

$$c_s = \left(\frac{\gamma k T_0}{m}\right)^{1/2} \ ,$$

where m is the mass of the fluid particle. c_s is not only the wave speed, but close to the mean speed of the fluid particles.

This was *linear theory.* The perturbations ρ_1 do not have a significant effect on the properties of the gas, and the waves do not change shape. But if the waves are larger, the terms ρ_1^2, etc., cannot be ignored, and the wave significantly changes the properties of the gas it goes through. In particular, where ρ_1 is large, the temperature of the gas is raised, and c_s is larger, so the wave travels fastest at its crest.



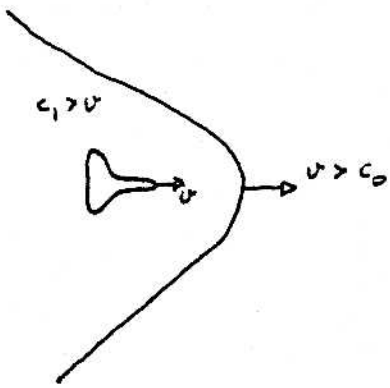
Because large waves tend to "break," a region develops where at a given x the density $\rho(x)$ attempts to become multivalued. But the density (or pressure, or temperature) cannot become multivalued. So something strange must happen when the gradient of ρ_1 (or P_1, or T_1, or u_1) becomes infinite.

We call a region where something changes very fast a *shock front*. In such regions, the continuum approximation about the gas breaks down – things are changing a lot on a scale of one mean free path – so the fluid equations are not valid.

Shocks are a tracer of *supersonic motion*. In the example above, the top of the strong wave is traveling faster than the speed of sound in the undisturbed medium. This is why a shock develops.

We say a shock occurs because the unperturbed (preshock) gas cannot be told (by "messages" traveling only at the speed of sound) to get out of the way of what is approaching. So the gas is hit by something without advance warning and is compressed a lot before it heats up enough that it can flow (subsonically) out of the way (subsonic with the new, higher, sound speed).

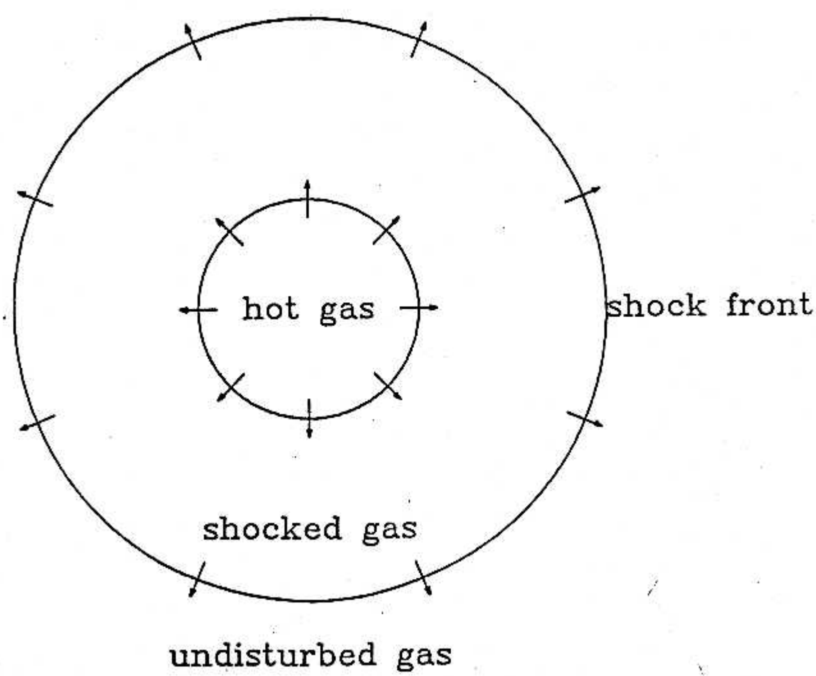
Supersonic plane



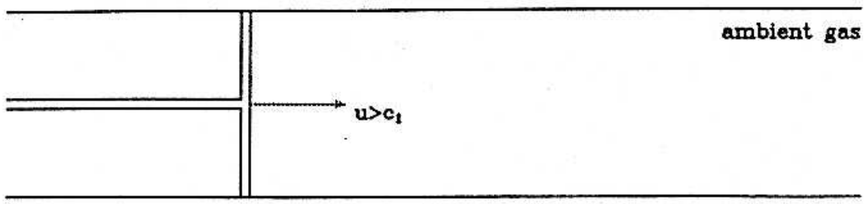
sonic boom is sound of bow shock passing

this is a "standoff shock" because it stands off from the front of the projectile

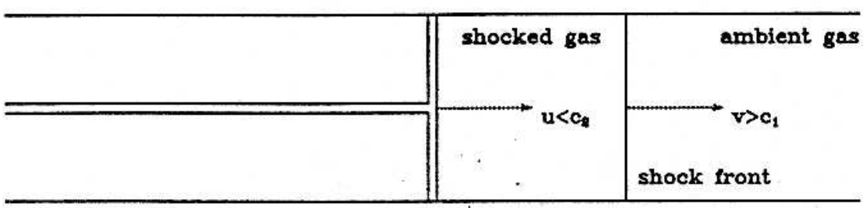
Explosion



Piston supersonically moving



t = 0



t > 0

The shocked gas is heated, may be dissociated (if the ambient gas is molecular), and has sound speed

$$c_2 = \left(\frac{\gamma_2 k T_2}{m_2}\right)^{1/2} \quad .$$

The shocked gas region is made up of all the stuff compressed between the shock front and the piston.

The Mach number of the shock is

$$M = \frac{v}{c_1} > 1 \quad ,$$

where the shock moves *supersonically* into the ambient gas. But

$$\frac{v}{c_2} < 1 \quad ,$$

and the shock moves *subsonically* relative to the shocked gas. The shocked gas "knows about" the piston and shock and comes to a steady state (well, almost, since it is still accumulating mass).