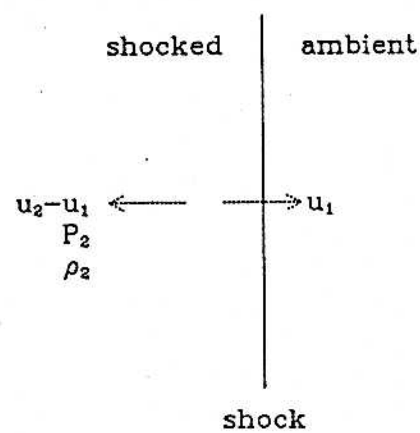
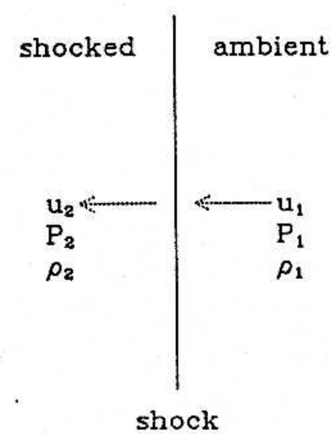
## 4.3. SHOCK JUMP CONDITIONS

Strangely enough, although we cannot describe the way that fluid behaves *inside* a shock (i.e., in the few mean free paths of the shock), we *can* say how the fluid conditions differ from side to side of the shock, i.e., as a result of the shock passing. For most purposes, this is sufficient.

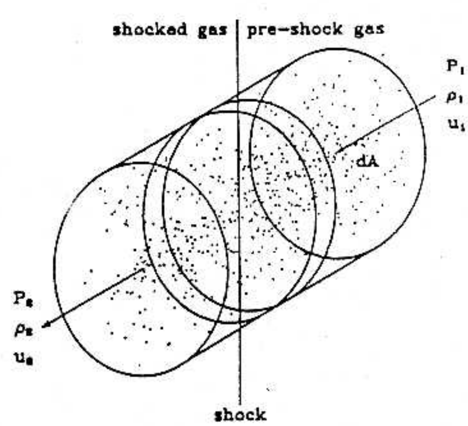
Let's find the jump conditions for a steady shock:




reference frame of ambient gas




reference frame of shock




The mass contained in the cylinder is constant if the shock is steady. So the mass flux into the shock is equal to the mass flux out. Hence,

$$\rho_1 u_1 = \rho_2 u_2 \quad .$$

The momentum density contained in the cylinder is also constant. Therefore, the net pressure force on the cylinder + the net momentum flux into the cylinder must be zero. Thus,

$$(P_1 dA - P_2 dA) + (\rho_1 u_1^2 dA - \rho_2 u_2^2 dA) = 0 \ ,$$

and so

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \ .$$

Finally, the energy contained in the cylinder is a constant. So the net flow of energy into the cylinder + work done on a gas in the cylinder is equal to zero. Therefore,

$$\left[u_1\left(\rho_1 e_1 + \frac{1}{2}\rho_1 u_1^2\right) dA - u_2\left(\rho_2 e_2 + \frac{1}{2}\rho_2 u_2^2\right) dA\right] + (P_1 u_1 - P_2 u_2) dA = 0 \ .$$

Thus,

$$u_1\left[P_1 + \rho_1\left(e_1 + \frac{1}{2}u_1^2\right)\right] = u_2\left[P_2 + \rho_2\left(e_2 + \frac{1}{2}u_2^2\right)\right] \ .$$

Hence,

$$e_1 + \frac{1}{2}u_1^2 + \frac{P_1}{\rho_1} = e_2 + \frac{1}{2}u_2^2 + \frac{P_2}{\rho_2} \ ,$$

using the first condition.

Note that the entropy per unit mass of the material entering the shock is much lower than that of the gas leaving the shock: the shock produces a great disorganization of the gas.

We can find this result in a different way by setting $\frac{\partial}{\partial t} = 0$ in the fluid equations derived earlier. From the equation of continuity,

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0 \quad \longrightarrow \quad \frac{\partial}{\partial x}(\rho u_x) = 0$$

$$\text{so} \quad \rho_1 u_1 = \rho_2 u_2 \ .$$

From Euler's equation,

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x \frac{\partial u_x}{\partial x}\right) = -\frac{\partial P}{\partial x} \quad \longrightarrow \quad \rho u_x \frac{\partial u_x}{\partial x} = -\frac{\partial P}{\partial x} \ .$$

But

$$\frac{\partial}{\partial x}(\rho u_x) = 0 \ ,$$

so

$$\frac{\partial}{\partial x}(P + \rho u_x^2) = 0 \ ,$$

and hence

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \ .$$

Finally, from the energy equation,

$$\frac{\partial}{\partial t}\left[\rho\left(e+\frac{1}{2}u_x^2\right)\right]=\rho\epsilon-\frac{\partial}{\partial x}\left[\rho u_x\left(e+\frac{1}{2}u_x^2+\frac{P}{\rho}\right)\right]\longrightarrow\frac{\partial}{\partial x}\left[\rho u_x\left(e+\frac{1}{2}u_x^2+\frac{P}{\rho}\right)\right]=0\,.$$

But

$$\frac{\partial}{\partial x}(\rho u_x)=0\ ,$$

so

$$e_1+\frac{1}{2}u_1^2+\frac{P_1}{\rho_1}=e_2+\frac{1}{2}u_2^2+\frac{P_2}{\rho_2}\ .$$

These three equations

$$\boxed{\begin{aligned}&\rho_1u_1=\rho_2u_2\\&P_1+\rho_1u_1^2=P_2+\rho_2u_2^2\\&e_1+\frac{1}{2}u_1^2+\frac{P_1}{\rho_1}=e_2+\frac{1}{2}u_2^2+\frac{P_2}{\rho_2}\end{aligned}}$$

are the Rankine–Hugoniot relations.

We assume that a gas passing through a shock is not dissociated: both $\gamma$ and $m$ (the mass of the molecule) remain unchanged. Then changes occur in pressure, density, and velocity. Recall, from the earlier notes on thermodynamics, that

$$\begin{aligned}e&=c_VT\\&=\frac{1}{\gamma-1}\cdot\frac{P}{\rho}\ ,\end{aligned}$$

and we replace $\frac{P}{\rho}$ using the speed of sound: $c^2=\frac{\gamma P}{\rho}$. Then we can rewrite the Rankine-Hugoniot conditions as:

$$\frac{\rho_2}{\rho_1}=\frac{u_1}{u_2}$$

$$u_1\left(1+\frac{c_1^2}{\gamma u_1^2}\right)=u_2\left(1+\frac{c_2^2}{\gamma u_2^2}\right)$$

$$\frac{1}{2}u_1^2+\frac{c_1^2}{\gamma-1}=\frac{1}{2}u_2^2+\frac{c_2^2}{\gamma-1}$$

The last equation is meaningless if $\gamma=1$, but correct if we multiply by $(\gamma-1)$, when it becomes $c_1^2=c_2^2$, i.e., $T_1=T_2$.

Define the compression ratio achieved by the shock

$$\frac{\rho_2}{\rho_1} = \psi^{-1}$$

so that $\psi$ is the *inverse* compression ratio.

Define the Mach number of the shock,

$$\mathcal{M} = \frac{u_1}{c_1} = \frac{\text{speed of shock}}{\text{speed of sound in unperturbed medium}} \quad ,$$

which is a measure of the strength of the shock. With these definitions,

$$u_1 = c_1\,\mathcal{M}$$
$$u_2 = c_1\,\psi\,\mathcal{M} \quad .$$

Substitute, then

$$\left(1 + \frac{1}{\gamma\mathcal{M}^2}\right) = \psi\left(1 + \frac{c_2^2}{c_1^2}\cdot\frac{1}{\gamma\psi^2\mathcal{M}^2}\right)$$
$$\frac{1}{2}\mathcal{M}^2 + \frac{1}{\gamma-1} = \frac{1}{2}\psi^2\mathcal{M}^2 + \frac{1}{\gamma-1}\cdot\frac{c_2^2}{c_1^2} \quad ,$$

which can be rewritten as two expressions for $\frac{c_2^2}{c_1^2}$ :

$$\frac{c_2^2}{c_1^2} = \psi + \gamma\,\psi\,\mathcal{M}^2(1-\psi)$$
$$= 1 + \frac{\gamma-1}{2}\,\mathcal{M}^2(1-\psi^2) \quad .$$

So, combining these equations produces an equation for $\psi$:

$$(\psi - 1) + \gamma\,\psi\,\mathcal{M}^2(1-\psi) - \frac{\gamma-1}{2}\,\mathcal{M}^2(1-\psi^2) = 0 \quad .$$

Factor out $(\psi - 1)$. $\psi \neq 1$, because this corresponds to no shock! ($\rho_1 = \rho_2$, $u_1 = u_2$, $P_1 = P_2$.) Then,

$$1 - \gamma\,\psi\,\mathcal{M}^2 + \frac{\gamma-1}{2}\,\mathcal{M}^2(\psi+1) = 0$$

i.e.,

$$\psi = \frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\cdot\frac{1}{\mathcal{M}^2} \quad .$$

We can now substitute in to find the jump conditions:

$$\frac{\rho_1}{\rho_2} = \frac{u_2}{u_1} = \frac{\gamma-1}{\gamma+1} + \frac{2}{\gamma+1}\cdot\frac{1}{\mathcal{M}^2}$$

$$\frac{P_2}{P_1} = \frac{2\gamma}{\gamma+1}\mathcal{M}^2 - \frac{\gamma-1}{\gamma+1}$$

$$\frac{T_2}{T_1} = \frac{c_2^2}{c_1^2} = \frac{P_2}{P_1}\cdot\frac{\rho_1}{\rho_2} = \left(\frac{2\gamma}{\gamma+1}\mathcal{M}^2 - \frac{\gamma-1}{\gamma+1}\right)\left(\frac{\gamma-1}{\gamma+1} + \frac{2}{1+\gamma}\cdot\frac{1}{\mathcal{M}^2}\right)$$

Notice that everything depends on $\mathcal{M}$, the Mach number, which measures the strength of the shock. $\gamma$, the ratio of specific heats, specifies the way that energy is shared between the internal energy and the kinetic energy of the flow.

For a very strong shock, $\mathcal{M} \to \infty$, and

$$\frac{\rho_1}{\rho_2} = \frac{u_2}{u_1} \longrightarrow \frac{\gamma-1}{\gamma+1}$$

$$\frac{P_2}{P_1} \to \infty \ .$$

For different types of gas, the maximum compression ratio varies:

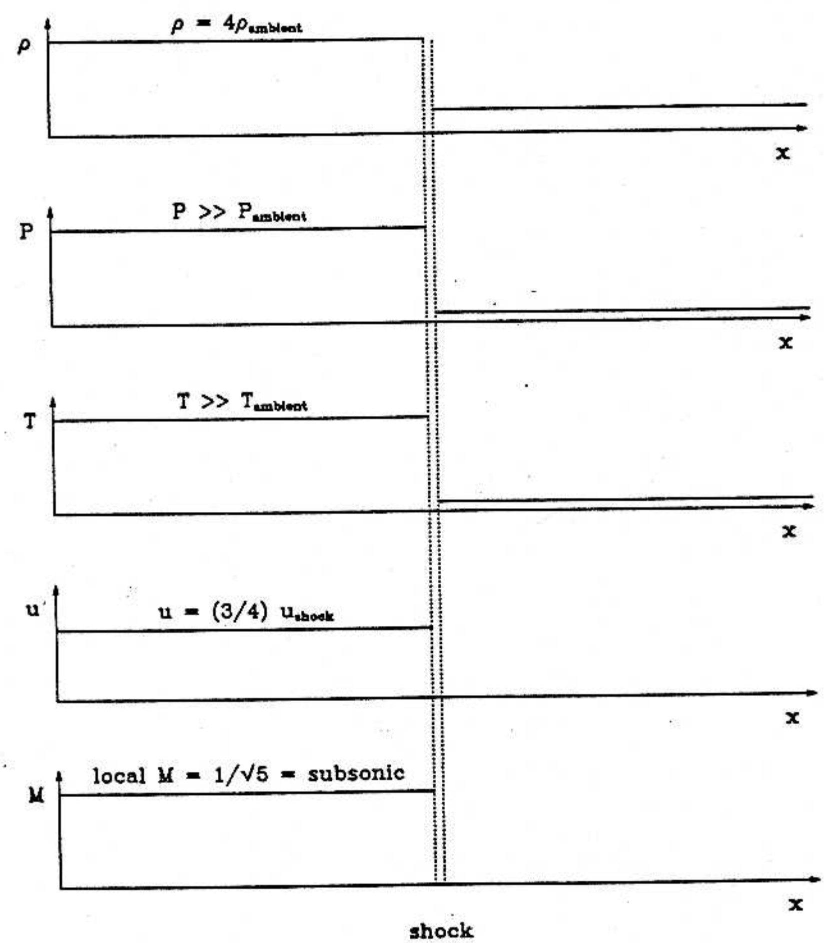
$\gamma = \frac{5}{3}$ $\quad \frac{\rho_1}{\rho_2} \to \frac{1}{4}$ maximum compression ratio is 4

$\gamma = \frac{4}{3}$ $\quad \frac{\rho_1}{\rho_2} \to \frac{1}{7}$ maximum compression ratio is 7

$\gamma = 1$ $\quad \frac{\rho_1}{\rho_2} \to \frac{1}{\mathcal{M}^2}$ maximum compression ratio is $\mathcal{M}^2$ (and $T_1 = T_2$)

But $\frac{P_2}{P_1}$ can become arbitrarily large.

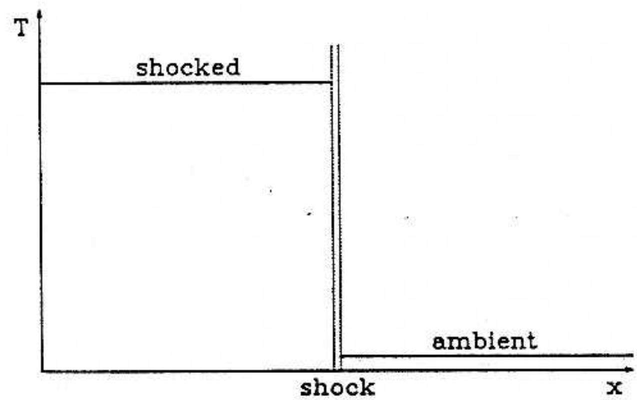
What do these look like for a gas with $\gamma = \frac{5}{3}$ (in the rest frame of the ambient medium)?



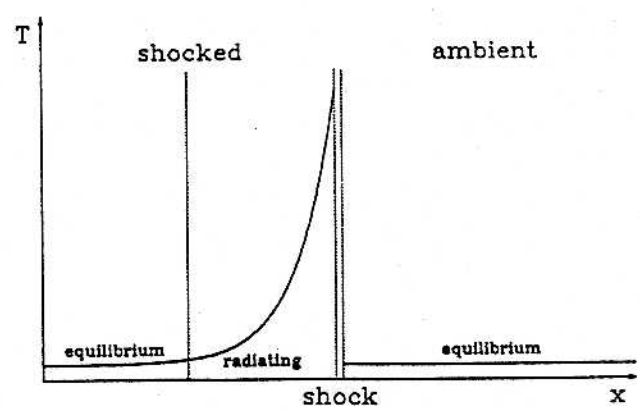
## 4.4. SHOCK STRUCTURES

Two limiting forms of shock are described in astrophysics: *adiabatic* shocks and *isothermal* (or *radiative*) shocks.

a) *Adiabatic* – The gas starts cool, usually, and does not radiate much energy in the shock front itself. Then, the equations we used are adequate and we get a thin shock (~ 1 mean free path in thickness) with the gas much hotter on one side than the other.




b) *Isothermal* – The gas starts in radiative equilibrium (usually fairly hot). In the first mean free path of molecules in the shock, we get a similar structure as before, but the gas now becomes hot enough to radiate energy efficiently (as optical line radiation, synchrotron radiation, ...). The temperature of the post-shock gas falls back to its equilibrium value *quickly* (many mean free paths, but not many AU, so we cannot see the internal structure of the shock).



It is common to describe the width of the cooling region plus the genuine shock region as the width of a *radiative* or *isothermal* shock. Then, we can apply jump conditions across the region as a whole, except we cannot use conservation of energy! The three equations are:

$$\rho_1 u_1 = \rho_2 u_2 \qquad \text{mass conservation}$$

$$P_1 + \rho_1 u_1^2 = P_2 + \rho_2 u_2^2 \qquad \text{momentum conservation}$$

$$T_1 = T_2 \qquad \text{constant temperature}$$

The third equation is equivalent to $c_1^2 = c_2^2$, the same as we get from the previous energy equation at $\gamma = 1$, i.e., previous jump conditions are correct at $\gamma \to 1$. Isothermal shocks are important because they permit very large compressions $\left(\frac{\rho_2}{\rho_1}\right)$.

### 4.4.1. What Produces Shocks in Astrophysics?

a) *Explosions*: Examples are supernova blast waves; impacts of asteroids on planets.

b) *Supersonic motions*: Examples are relativistic fluid flows from active galactic nuclei; shocks ahead of cometary nuclei; shocks ahead of stars; spiral density waves. Supersonic motions occur anywhere that there is a *fast* process involving a gas; anywhere there is a lot of energy being released; and almost anywhere a gas is expanding or contracting.

In brief, shocks and shock phenomena are very common in astrophysics, and a lot of things we do not understand are customarily ascribed to shock phenomena.

### 4.4.2. Characteristics of Shocks

The temperature of the interstellar medium (ISM) is $T \sim 10^4$ K. Hence, the sound speed, $c_s = \left(\frac{\gamma k T}{m}\right)^{1/2} \sim 12 \text{ km s}^{-1}$.

Stars have random velocities $\sim 15 \text{ km s}^{-1}$; therefore, stellar bow shocks exist and have Mach number $\mathcal{M} \sim 1.2$ (weak shocks).

Some classes of stars emit fast stellar winds, at $\sim$ 100–3000 km s$^{-1}$; hence, the winds meet the ISM in bow shocks with $\mathcal{M} =$ 8–250 (strong shocks).

Supernova ejecta have velocities $\sim 10^4 \text{ km s}^{-1}$. Therefore, the ejecta meet the ISM in shocks with $\mathcal{M} \sim 10^3$ (very strong).

Consider a supernova shock with $\mathcal{M} \sim 10^3$ and $\gamma = \frac{5}{3}$. Thus, the jump conditions are

$$\frac{\rho_2}{\rho_1} = 4$$
$$\frac{u_2}{u_1} = \frac{1}{4}$$
$$\frac{P_2}{P_1} = 1.3 \times 10^6$$
$$\frac{T_2}{T_1} = 3 \times 10^5$$

Thus, the heated gas behind the shock has a temperature $T_2 \sim 3 \times 10^9$ K – and it will emit radiation as a very hard x-ray source. This shocked gas is much hotter than the gas in the center of the Sun. (But there is no nucleosynthesis because the density is very low.)