4.5. STRONG SPHERICAL SHOCKS (SUPERNOVAE)

There are four phases in the history of a supernova:

a) *Free expansion* (while the mass of ejecta exceeds the mass of the ISM swept up)

b) *Sedov phase* (adiabatic phase)

c) *Snowplow phase* (isothermal shock)

d) *Decay phase* (subsonic)

a) *Free expansion* (the earliest part of the expansion)

This occurs while the mass swept up in the expansion is less than the ejected mass so that the ejected material has not shared its momentum with much other mass, so it is almost undecelerated and the velocity of expansion is constant.

Define:

r_s = radius of shock front

v_e = velocity of shock front at initial ejection

M_e = mass of ejecta

ρ_a = density of ambient medium

Then, at time t after the explosion,

$$r_s = v_e t \quad ,$$

$$v_e = \text{constant} \quad .$$

This continues while

$$M_e > \frac{4\pi r_s^3}{3}\rho_a \quad ,$$

i.e., until a time

$$\tau_1 \sim \left(\frac{3M_e}{4\pi\rho_a v_e^3}\right)^{1/3} \quad .$$

If the kinetic energy in the ejection is E_e, then v_e and E_e are related by

$$E_e = \frac{1}{2}M_e v_e^2 \quad ,$$

and the free expansion phase ends at time

$$\tau_1 \simeq \left(\frac{3}{8\pi\sqrt{2}}\right)^{1/3} M_e^{5/6}\rho_a^{-1/3}E_e^{-1/2} \quad .$$

If we suppose that all the energy of the explosion goes initially into the kinetic energy of the ejecta, it is clear that this kinetic energy is gradually shared with material flowing through the shock front and will be converted partially to thermal energy inside the shock.

During the free expansion phase, the (small) mass of material entering the shock does not detract much from the energy; it is merely heated to

$$\frac{T_2}{T_1} \approx \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} \mathcal{M}^2 \qquad \text{(from jump conditions)}$$

if $\mathcal{M} \gg 1$. But $\mathcal{M} = \frac{v_e}{c_s} = v_e \left(\frac{\mu}{\gamma k T_1}\right)^{1/2}$, so

$$\frac{T_2}{T_1} = \frac{2\gamma(\gamma-1)}{(\gamma+1)^2} v_e^2 \cdot \frac{\mu}{\gamma k T_1} ,$$

and hence

$$T_2 = \frac{2(\gamma-1)}{(\gamma+1)^2} \cdot \frac{\mu v_e^2}{k} .$$

This is cooler than the initial temperature of the gas in the explosion. So the gas inside the shock tends to cool to temperature T_2 as the free expansion phase continues.

The detailed structure of the flow behind the shock *depends strongly* on the initial conditions during this phase, since there has been insufficient time for the flow pattern to smooth itself out.

For our canonical explosion,

$$E_e = 10^{51} \text{ erg}$$
$$M_e = 0.5\ M_\odot = 10^{33} \text{ g}$$
$$\rho_a = 2 \times 10^{-24} \text{ g cm}^{-3} .$$

The initial velocity of the ejecta is

$$v_e \simeq \left(\frac{2E_e}{M_e}\right)^{1/2} \sim 10^4 \text{ km s}^{-1} ,$$

and this phase lasts until $t = \tau_1$,

$$\tau_1 \simeq 100 \text{ yr}$$

after the explosion, at which time the shock radius $r_s \simeq 2$ pc.

b) <u>*Sedov phase*</u>

During this phase, the external medium is dynamically important (the mass in the ejecta is now much less than the mass swept up from the ISM). The shock front decelerates, so the ejecta behind the shock front tend to catch up and pile more mass into a thin shell. This is *not* due to the pressure of ambient medium, which exerts an insignificant force on the shock, but is merely due to the sharing of momentum between the initial ejecta and the swept-up gas.

At this stage, the rate of radiation of energy from the shock is relatively low; the gas is too hot to lose radiation efficiently, and we develop an adiabatic shock front.

The shock is still very strong: the pressure behind the shock is much greater than the pressure ahead of the shock.

$$\frac{P_2}{P_1} \sim \frac{2\gamma}{\gamma+1}\mathcal{M}^2 \gg 1 \ .$$

So we can neglect P_1 by comparison with P_2, and the nature of the flow is determined entirely by the energy of the explosion and the density of the ambient medium.

The pressure behind the shock, P_2, is approximately equal to the energy density inside the shock, $\frac{E}{\frac{4}{3}\pi r_s^3}$. Then the jump conditions for $\mathcal{M} \gg 1$ give

$$\frac{P_2}{P_1} = \frac{2\gamma}{\gamma+1}\mathcal{M}^2 \ ,$$

i.e.,

$$P_2 = \frac{2\gamma}{\gamma+1} \cdot \frac{u_1^2}{c_1^2} P_1 = \frac{2}{\gamma+1} \cdot u_1^2 \rho_1 \ ,$$

since $c_1^2 = \frac{\gamma P_1}{\rho_1}$. Thus,

$$u_1^2 = \frac{(\gamma+1)P_2}{2\rho_1} \sim \frac{3(\gamma+1)E}{8\pi\rho_1 r_s^3} \ ,$$

i.e., the shock velocity

$$\frac{dr_s}{dt} = \left[\frac{3(\gamma+1)E}{8\pi\rho_1}\right]^{1/2} r_s^{-3/2} \ .$$

This can be integrated:

$$\int r_s^{3/2} dr_s = \int dt \left[\frac{3(\gamma+1)E_e}{8\pi\rho_a}\right]^{1/2} \ ,$$

i.e.,

$$r_s(t) = \left(\frac{5}{2}\right)^{2/5} \left[\frac{3(\gamma+1)E_e}{8\pi\rho_a}\right]^{1/5} t^{2/5} \quad ,$$

i.e.,

$$\boxed{r_s(t) \simeq 1.15\, E_e^{1/5} \rho_a^{-1/5} t^{2/5} \quad ,} \qquad \gamma = \tfrac{5}{3}$$

and the temperature immediately behind the shock is

$$T_2 = \frac{P_2}{\rho_2} \cdot \frac{\mu}{k} = \frac{2\rho_1 u_1^2}{\gamma+1} \cdot \frac{1}{\rho_2} \cdot \frac{\mu}{k} \quad .$$

But $\dfrac{\rho_2}{\rho_1} = \dfrac{\gamma+1}{\gamma-1}$, and so

$$T_2 = \frac{2(\gamma-1)}{(\gamma+1)^2} \cdot \frac{\mu}{k} \cdot \left(\frac{dr_s}{dt}\right)^2 \quad ,$$

where μ is the mass per molecule. Using the result for $r_s(t)$, this can be written

$$\boxed{T_2 = 0.040 \left(\frac{\mu}{k}\right) E_e^{2/5} \rho_a^{-2/5} t^{-6/5} \quad .} \qquad \gamma = \tfrac{5}{3}$$

Of course, we can do better than this!

We see that the only dimensional quantities in the problem are r, t, ρ_a, and E_e. All of the other variables depend only on these. But there is only one dimensionless parameter that can be formed from r, t, ρ_a, and E_e:

$$\xi = r\left(\frac{\rho_a}{E_e t^2}\right)^{1/5} \quad .$$

So the solution must depend only on ξ.

This is called a *similarity solution* and implies that the initial conditions of the explosion have been forgotten by this phase of the expansion. This tells us at once that

$$r_s = (\text{constant})\rho_a^{-1/5} E_e^{1/5} t^{2/5} \quad ,$$

just as I derived.

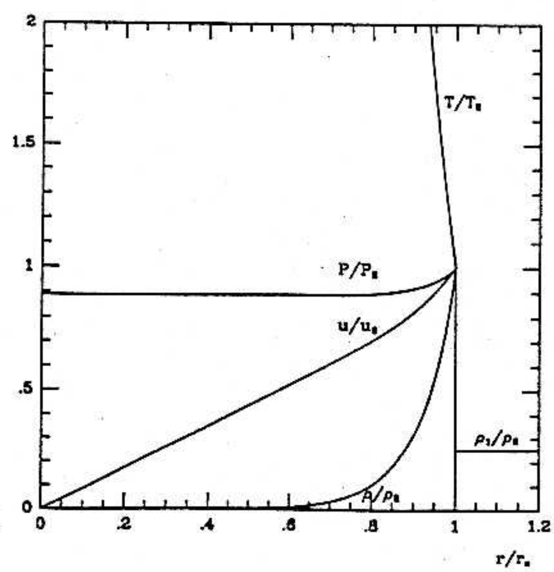
Now, we solve the equations of motion of the gas *fully* to get the actual conditions inside the shock shell. For a spherical expansion, the equations of fluid motion become

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 \rho v) = 0$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial P}{\partial r}$$
$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial r}\right)\log(P\rho^{-\gamma}) = 0 \quad ,$$

and our search for a similarity solution leads us to substitute

$$\rho = \rho_a \cdot f_1(\xi)$$
$$P = \rho_a \cdot \frac{r^2}{t} \cdot f_2(\xi)$$
$$v = \frac{r}{t} f_3(\xi) \quad ,$$

where f_1, f_2, and f_3 are dimensionless functions, and the terms ρ_a, $\rho_a \cdot \frac{r^2}{t}$, and $\frac{r}{t}$, which precede f_1, f_2, and f_3, make the expression dimensionally correct. The solution is a bit involved (Landau and Lifshitz, *Fluid Mechanics*, 1st edition, pp. 392–396 – a book astrophysicists swear by, or, more frequently, *at*!), but it leads to ρ, P, and v as functions of the radius inside the shock. We get:



T_2 = temperature just behind shock	$= \frac{2(\gamma-1)}{(\gamma+1)^2} \cdot \frac{\mu}{k} \cdot \dot{r}_s^2$	$\propto t^{-6/5}$
P_2 = pressure just behind shock	$= \frac{2}{\gamma+1} \rho_1 \dot{r}_s^2$	$\propto t^{-6/5}$
ρ_2 = density just behind shock	$= \frac{\gamma+1}{\gamma-1} \rho_1$	$\sim$ constant
v_2 = velocity just behind shock	$= \dot{r}_s - u_2 = \frac{2}{\gamma+1} \cdot \dot{r}_s$	$\propto t^{-3/5}$

Most of the mass in the shock is at $r \sim r_s$ (the shock has developed a dense shell), and the hottest part of the shocked region is at the center.

c) *Snowplow phase*

Eventually, as T_2 drops ($\propto t^{-6/5}$) to $\sim 10^6$ K, radiation by the dense material behind the shock becomes important.

The thin dense layer behind the shock radiates energy rapidly (mostly as optical line emission – this is the phase when the SNR is most obvious, optically, after the first flash). The shock becomes isothermal. Now, the shock's dynamics are determined entirely by mass and momentum conservation – we say the shock slows down by sweeping up mass: the snowplow phase.

If M_{sp} and v_{sp} are the mass in the shock and the shock velocity when this phase starts, then at some later time, the mass in the shock and shock velocity obey

$$Mv = M_{sp} v_{sp} \quad ,$$

by momentum conservation. Mass in the ejecta is much less than either M_{sp} or M; M and M_{sp} are predominantly matter swept up from the ISM. So

$$M = \frac{4}{3}\pi r_s^3 \rho_a \quad ,$$

i.e.,

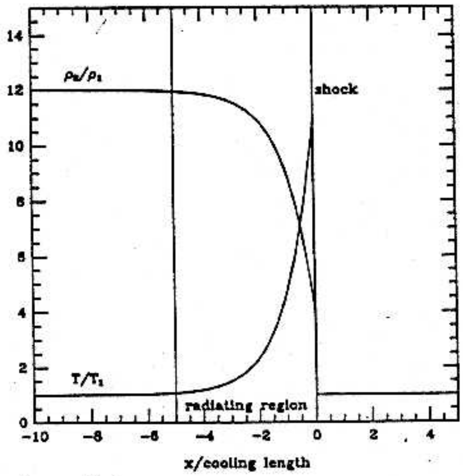
$$\frac{4}{3}\pi r_s^3 \rho_a \frac{dr_s}{dt} = M_{sp} v_{sp} \quad .$$

Integrate:

$$r_s(t) = \left(\frac{3M_{sp}v_{sp}}{\pi\rho_a}\right)^{1/4} (t - t_{sp})^{1/4} \propto t^{1/4} \quad ,$$

where $t \gg t_{sp}$. The expansion is slower here than in the Sedov phase and decelerates faster $\longrightarrow \dot{r}_s \propto t^{-3/4}$.

In the snowplow phase, we have *high compression* "in the shock." This high compression of the ISM may do interesting things (such as compress clouds enough for them to collapse to form a new generation of stars).



d) *Decay phase* (subsonic)

When $\dot{r}_s$ drops to the ISM sound speed ~ 12 km s^{-1}, we no longer have a shock. The SNR dissipates, since random motions in the ISM destroy it.

To summarize, the phases of expansion are:

Free phase:	$r_s \propto t$	$T_2 \sim$ constant
Sedov phase:	$r_s \propto t^{+2/5}$	$T_2 \propto t^{-6/5}$
Snowplow phase:	$r_s \propto t^{1/4}$	$T_2 \propto t^{-3/2}$
Subsonic phase:	$r_s \sim$ constant	$T_2 \sim T_1$

e) *Transition from Sedov phase to snowplow phase*

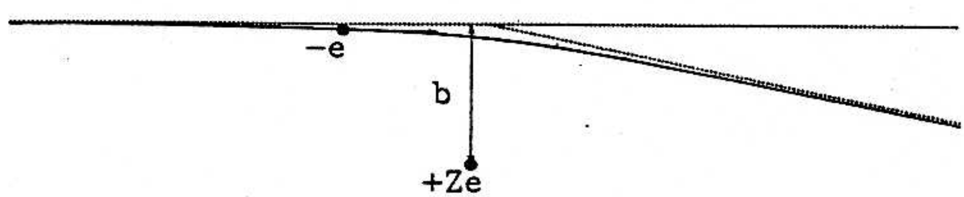
This occurs when the cooling time for gas passing through the shock is comparable with the quantity $\frac{r_s}{(u_2 - u_1)}$, the time it takes this gas to go a sensible fraction of the distance across the SNR. But what is the cooling time?

Consider a blob of hot gas. How will it lose energy?

— by free–free radiation ... bremsstrahlung ... if $T \gtrsim 10^6$ K.

— by line radiation ... if $T \lesssim 10^6$ K.

We are concerned with gas cooling from above $\sim 10^6$ K, so we can see *very roughly* where the Sedov phase ends by considering only bremsstrahlung cooling. So we need to calculate the energy loss rate in bremsstrahlung.



$$\text{acceleration of electron, } a \simeq \frac{Ze^2}{b^2} \cdot \frac{1}{m}$$

$$\text{energy loss rate } \sim \frac{2e^2 a^2}{3c^3}$$

$$\text{time while losses are operative} \sim \frac{b}{v}$$

Therefore, the energy loss per encounter is

$$\Delta E \sim \frac{2}{3} \cdot \frac{e^2}{c^3} \cdot \left(\frac{Ze^2}{mb^2}\right)^2 \frac{b}{v} \ .$$

The number of encounters/unit time/unit volume $= n_e n_Z v \,.\, 2\pi b \, db$ in an annulus of radius b. Hence, the total rate of loss of energy/unit volume, Λ_{ff}, is

$$\Lambda_{ff} = \int_{b_{min}}^{\infty} n_e n_Z v \,.\, 2\pi b \, db \,.\, \frac{2}{3} \cdot \frac{e^2}{c^3} \cdot \frac{Z^2 e^4}{m^2 b^4} \cdot \frac{b}{v}$$

$$\simeq \frac{Z^2 n_e n_Z e^6}{m^2 c^3} \cdot \frac{1}{b_{min}} \cdot \left(\frac{4\pi}{3}\right) \quad ,$$

where b_{min} is the minimum value of $b \sim \frac{h}{mv}$, the Compton wavelength of electron, and for the velocity we use $v \sim \left(\frac{3kT}{m}\right)^{1/2}$, the mean speed of thermal particles. Then

$$\Lambda_{ff} \simeq \frac{Z^2 n_e n_Z e^6}{c^3 mh} \left(\frac{kT}{m}\right)^{1/2} \quad .$$

very rough

The exact result is

$$\Lambda_{ff} = \left(\frac{2\pi}{3}\right)^{1/2} \frac{32}{3} \overline{g}(T) \cdot \frac{Z^2 n_e n_Z e^6}{c^3 mh} \left(\frac{kT}{m}\right)^{1/2} \quad ,$$

where $\overline{g}$ is another Gaunt factor, some function that is usually of order unity. Numerically,

$$\Lambda_{ff} = 1.4 \times 10^{-27} Z^2 \left(\frac{n_e}{\text{cm}^{-3}}\right) \left(\frac{n_Z}{\text{cm}^{-3}}\right) \left(\frac{T}{\text{K}}\right)^{1/2} \overline{g} \ \text{erg cm}^{-3}\ \text{s}^{-1} \quad .$$

by QED

The free–free cooling time

$$\tau_{ff} = \frac{U}{\Lambda_{ff}} \quad ,$$

energy per unit vol. / energy loss rate / unit vol.

where U is the energy density of the gas $= \frac{3}{2} n_Z kT + \frac{3}{2} n_e kT$. So, with $Z = 1$ and $\overline{g} \sim 1$:

$$\tau_{ff} \simeq \frac{3nkT}{1.4 \times 10^{-27} n^2 T^{1/2}} \ \text{sec} \quad ,$$

where n is in cm^{-3} and T is in K, i.e.,

$$\tau_{ff} \sim 3 \times 10^{11} \left(\frac{T}{\text{K}}\right)^{1/2} \left(\frac{n}{\text{cm}^{-3}}\right)^{-1} \ \text{sec} \quad ,$$

which is proportional to $t^{-3/5}$, i.e., drops during expansion. Thus, the transition of phases occurs when

$$\tau_{ff} \lesssim \frac{r_s}{\dot{r}_s}$$

(since $\frac{r_s}{\dot{r}_s}$ is the characteristic timescale of expansion), i.e., when $t \sim \frac{2}{5}\tau_{ff}$. Use

$$n \simeq \frac{\rho_2}{\mu} \sim \frac{\gamma+1}{\gamma-1} \cdot n_a$$

$$T \simeq T_2 \sim 0.040\, E_e^{2/5} \rho_a^{-2/5} t^{-6/5} \left(\frac{\mu}{k}\right) \quad ,$$

then for our canonical explosion with $E_e = 10^{51}$ erg, $\rho_a \sim 2 \times 10^{-24}$ g cm^{-3}; $\mu \sim 1.67 \times 10^{-24}$ g, the transition is at $t = \tau_2$, and

$$\tau_2 \sim 8000 \ \text{yr} \quad .$$

overestimate; due to Λ_{ff} being underestimated by $\frac{1}{2}$ to $\frac{1}{6}$.