6

The depth dependence of the source function

6.1 Empirical determination of the depth dependence of the source function for the sun

As we have seen in Chapter 5, the intensity leaving the stellar surface under an angle ϑ is given by

$$I_\lambda(0, \vartheta) = \sum_i A_{\lambda i} \cos^i \vartheta \qquad \text{if } S_\lambda(\tau_\lambda) = \sum_i a_{\lambda i} \tau_\lambda^i, \tag{6.1}$$

where the relation between $A_{\lambda i}$ and $a_{\lambda i}$ is known to be (see (5.11))

$$A_{\lambda i} = a_{\lambda i} i! \tag{6.2}$$

If we can measure $A_{\lambda i}$ for a given wavelength λ, i.e., if we measure the center-to-limb variation of $I_\lambda(0, \vartheta)$, then we can determine the depth dependence of the source function. We can do this for all wavelengths λ and obtain $S_\lambda(\tau_\lambda)$ for all λ.

As an example, we have chosen the wavelength $\lambda = 5010$ Å. Measured data for the center-to-limb variation can be approximated by (see the table in Chapter 6, problem 1, page 233).

$$I_\lambda(0, \vartheta) = [a_0(\lambda) + a_1(\lambda) \cos \vartheta + 2a_2(\lambda) \cos^2 \vartheta] I_\lambda(0, 0). \tag{6.3}$$

From this expression we find, according to (6.1), that

$$S_\lambda(\tau_\lambda(5010\,\text{Å})) = [a_0(\lambda) + a_1(\lambda)\tau_\lambda(5010\,\text{Å}) + a_2(\lambda)\tau_\lambda^2(5010\,\text{Å})] I_\lambda(0, 0). \tag{6.4}$$

In Fig. 6.1 we have plotted the source function $S_\lambda(\tau_\lambda(5010\,\text{Å}))$ as a function of the optical depth according to equation (6.4). If we now go one step further and assume again that $S_\lambda = B_\lambda(\mathrm{T}(\tau_\lambda))$, then we know at each optical depth τ_λ how large the Planck function B_λ has to be. Since B_λ depends only on the temperature T and on the wavelength λ we have one equation at each optical depth to determine the temperature T, which will give the required value for the Planck function, i.e.,

$$S_\lambda(\tau_\lambda) = B_\lambda(T(\tau_\lambda)) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1}, \tag{6.5}$$

where, in our example, the wavelength λ is 5010 Å.

From equation (6.5) and the values of $S_\lambda(\tau_\lambda)$ derived from the center-to-limb variation, the temperature T can be determined as a function of the optical depth for the wavelength of 5010 Å.

The optical depth dependence of the temperature obtained in this way for the wavelength of 5010 Å is shown in Fig. 6.2. As we expected, the temperature increases with increasing optical depths.

6.2 The wavelength dependence of the absorption coefficient in the sun

There is, of course, nothing magic about the wavelength of 5010 Å. We can follow the same procedure for any other wavelength λ. For each wavelength we find a relation $T(\tau_\lambda)$, showing the dependence of the temperature on that particular optical depth for the chosen wavelength. In Fig. 6.3 we show the results for a number of wavelengths. We find different curves for the different wavelengths because κ_λ is dependent on λ. For a given layer in the solar atmosphere with a given geometrical depth

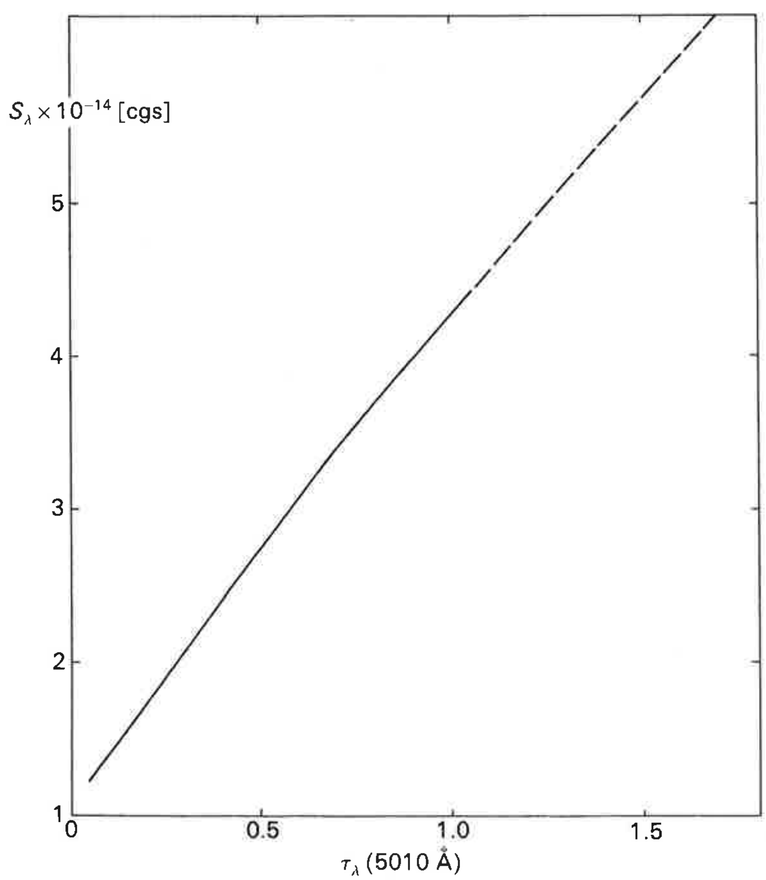

Fig. 6.1. The dependence of the source function S_λ (5010 Å) on the optical depth τ_λ for the wavelength of 5010 Å. From $\tau_\lambda > 1$ we receive little radiation. The values for deeper layers therefore become less certain as indicated by the dashed line.

t and a particular temperature, we have different optical depths, τ_λ, for different wavelengths λ. We know, however, that at a given geometrical depth t we must have *one* value for the temperature. All points with the same temperature T_0 must therefore refer to the same geometrical depth. We draw a horizontal line through Fig. 6.3 which connects all the points with the temperature $T_0 = 6300$ K. All these points must belong to the same geometrical depth t. We now write the optical depths τ_λ in the form

$$\tau_\lambda = \kappa_\lambda t. \tag{6.6}$$

Here κ_λ is an average over depth down to depth t. For all the points on the horizontal line we know that the value of t is the same. We read off at the abscissa the optical depths $\tau_{\lambda_1}, \tau_{\lambda_2}, \tau_{\lambda_3}$, etc., which belong to this depth t. Equation (6.6) now tells us that

$$t_{\lambda_1} = \frac{\tau_{\lambda_3}}{\kappa_{\lambda_3}} = \frac{\tau_{\lambda_2}}{\kappa_{\lambda_2}} = \frac{\tau_{\lambda_1}}{\kappa_{\lambda_1}}, \tag{6.7}$$

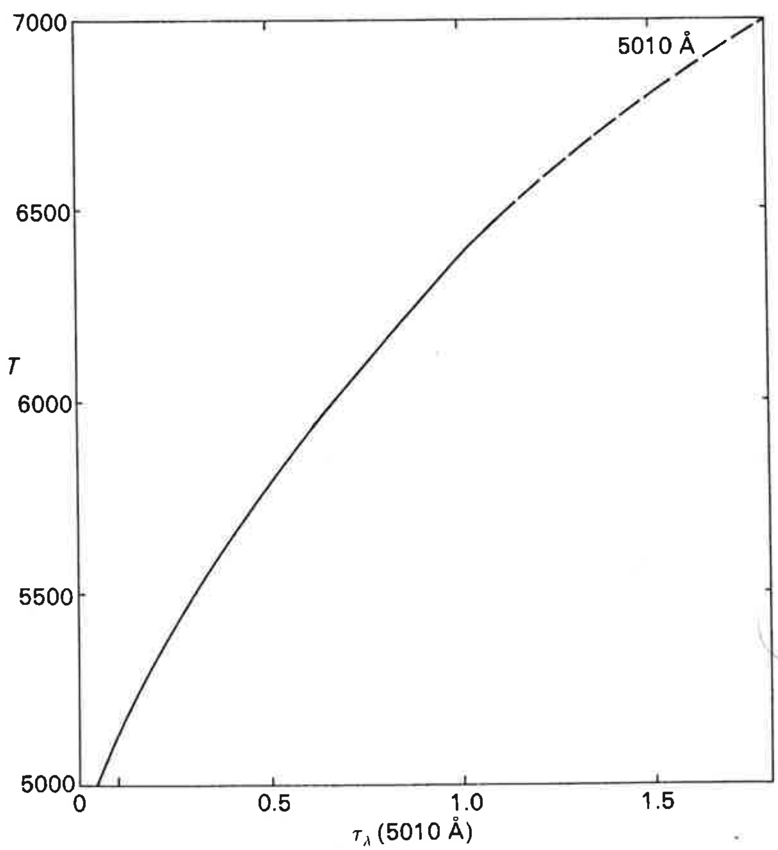

Fig. 6.2. The dependence of the solar temperature T on the optical depth at $\lambda = 5010$ Å is shown as derived from the observed center-to-limb variation of the solar intensity at $\lambda = 5010$ Å. For $\tau_\lambda > 1$ the values become less certain, as indicated by the dashed line.

or

$$\frac{\tau_{\lambda_3}}{\tau_{\lambda_2}} = \frac{\kappa_{\lambda_3}}{\kappa_{\lambda_2}} \quad \text{and} \quad \frac{\tau_{\lambda_3}}{\tau_{\lambda_1}} = \frac{\kappa_{\lambda_3}}{\kappa_{\lambda_2}}, \qquad 6.8$$

which means we can determine the ratio of all κ_{λ_i} with respect to one κ_{λ_0}.

For a given temperature $T(t)$ we can also plot the values $\tau_\lambda(T)$ as a function of λ. Because t is the same for all λ, the wavelength dependence of τ_λ shows directly the wavelength dependence of κ_λ, as was first derived by Chalonge and Kourganoff in 1946.

If κ_{λ_i} does depend on the geometrical depth t, we will find an average value of κ_{λ_i} down to the depth t. For different temperatures, i.e., for different depths, these averages will be somewhat different. The results are shown in Fig. 6.4.

It turns out that the wavelength dependence of κ_λ agrees with the absorption coefficient which was calculated for the negative hydrogen ion

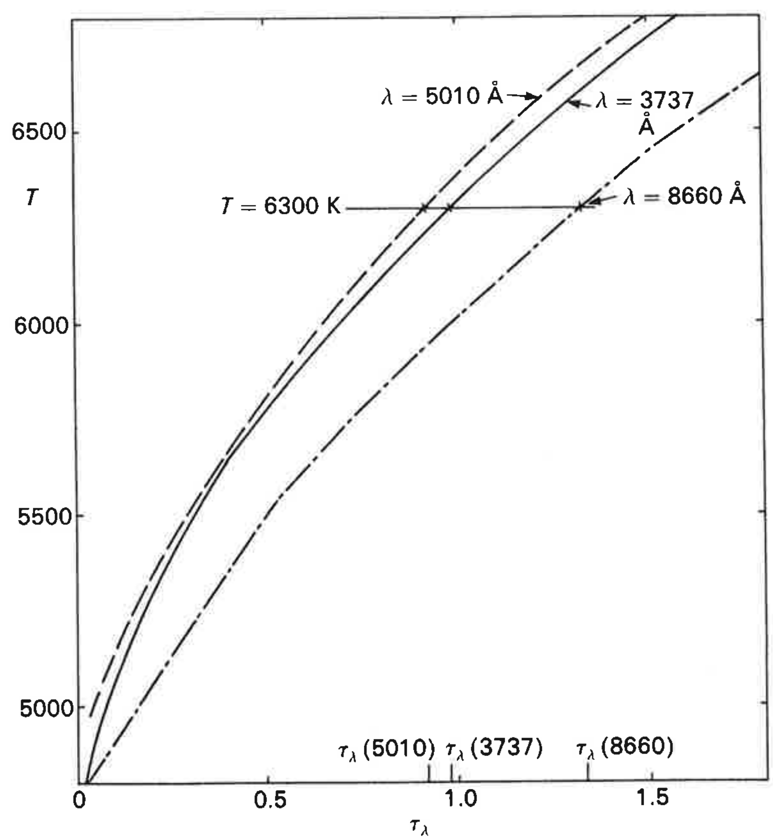

Fig. 6.3. The dependences of the temperature T on the optical depth $\tau_{\lambda_1}, \tau_{\lambda_2}, \tau_{\lambda_3}$, where $\lambda_1 = 5010$ Å, $\lambda_2 = 3737$ Å, $\lambda_3 = 8660$ Å. The horizontal line connects points of equal temperature $T_0 = 6300$ K. These points must all belong to the same geometrical depth t. The corresponding optical depths for the different wavelengths can be read off at the abscissa.

H^- (see Fig. 6.4). Rupert Wildt first suggested in 1938 that H^- might be most important for the continuous absorption coefficient in the solar photosphere. It is still very difficult to measure the absorption coefficient for H^- in the laboratory because it is so small. The solar photosphere needs a depth of roughly 100 km to reach an optical depth of 1. In the laboratory we can use higher densities, but the required path lengths are still very large. Therefore, it was very important when Chalonge and Kourganoff (1946) showed that the wavelength dependence of κ_λ could be measured for the sun and that it did agree with the one calculated for H^-.

6.3 Radiative equilibrium

We pointed out above that the energy transport in the star, which has to bring the energy from the interior to the surface, requires that the temperature decreases from the inside out, because a thermal energy flow goes only in the direction of decreasing temperature. This discussion indicates that the energy flux through the star is related to the temperature gradient. Here we will show that we can actually calculate the temperature gradient if we know how much energy is transported outwards at any given point and by which transport mechanism.

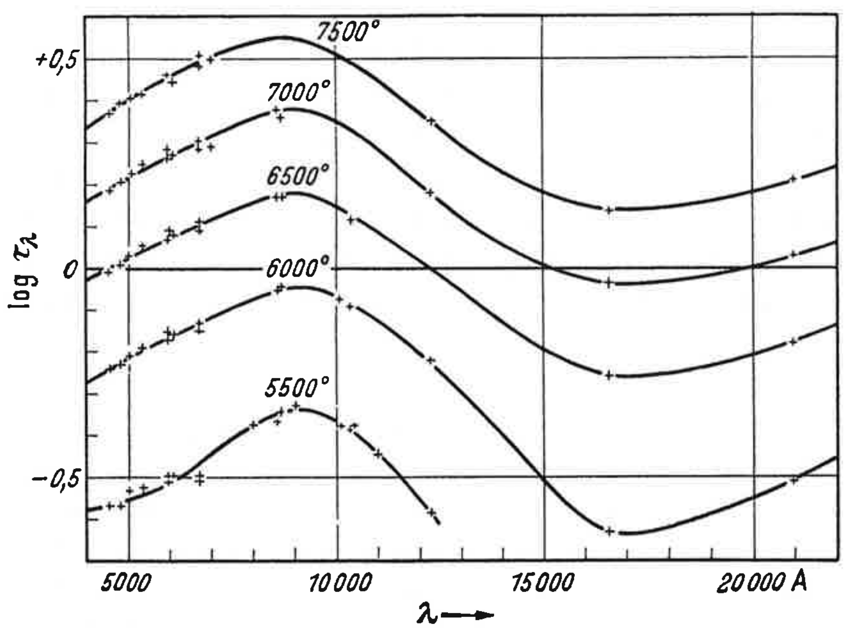

Fig. 6.4. The wavelength dependence of the continuous absorption coefficient κ_λ in the solar photosphere as determined by Chalonge and Kourganoff in 1946. (From Unsöld, 1955, p. 116.)

We know from observation how much energy leaves the surface of a star per cm^2 each second. We expect that a larger temperature gradient is needed for a larger energy transport. We also expect a larger temperature gradient if the heat transport is made difficult by some obstacle. For instance, if we think about a room with an outside wall where the heat is leaving, and with a radiator on the inside wall on the opposite side, then there will be a temperature gradient from the radiator wall to the outside wall. The temperature gradient will be larger if the window is opened on a cool day, which will cause a large energy flux to go out the window. If a large flux is to go out the window the temperature gradient will have to be larger if shelves are erected in front of the radiator which inhibit the heat flow. The heat flow is largest for the largest temperature gradient and for the smallest number of obstacles. We can turn this around and say if a large amount of heat energy transport is required we need a steep temperature gradient, and if the energy transport is made difficult, we also need a steep temperature gradient in order to transport a given amount of energy.

If we have measured the amount of energy leaving the stellar surface, how do we know how much energy is transported outwards at a deeper layer? We know that the radiation we see comes from a layer which has a geometrical height of a few hundred kilometers. We can easily calculate how much thermal energy is contained in this layer. For instance, for the sun the number of particles per cm^3 is about 10^{17}. The total number of particles in a column of 100 km height and a cross-section of $1\,cm^2$ is then roughly 10^{24}. Each particle has a thermal energy of $\frac{3}{2}kT$, which is about 10^{-12} erg. The total thermal energy of this column is therefore about $10^{12}\,erg\,cm^{-2}$. The radiative energy loss per cm^2 of the solar surface is equal to πF, which for the sun is measured to be $6.3 \times 10^{10}\,erg\,cm^{-2}\,s^{-1}$. The energy content of the solar photosphere therefore can only last for 15 seconds if the sun keeps shining at a constant rate. However, we see that the photosphere is not cooling off; therefore, we know that the amount of energy which the sun is losing at the surface must constantly be put into the photosphere at the bottom. If exactly the same amount of energy were not replenished at any given time the solar photosphere would very rapidly change its temperature. If too little energy were to flow in from the bottom the photosphere would cool off; if too much were transported into the bottom of the photosphere it would heat up and more energy would leave the solar surface. Nothing like this happens, so the amount of energy flowing into the photosphere per cm^2 each second must equal exactly the amount of energy lost at the surface, which is πF. The same argument can be made for the layer below the photosphere. It too must get the same amount of energy from the next deeper layer in order to remain at the same temperature. Therefore, we must have the situation shown in Fig. 6.5. **The energy flux must remain constant with depth.**

This means

$$\frac{\mathrm{d}F}{\mathrm{d}t} = 0, \qquad \text{or} \qquad \frac{\mathrm{d}F}{\mathrm{d}\tau} = 0$$

or

$$\pi F = \text{const.} = \sigma T_{\text{eff}}^4. \tag{6.9}$$

Equation (6.9) is a basic condition for a star if it is to remain constant in time and not supposed to heat up or cool off. We call this the *condition of thermal equilibrium*. This should not be confused with the condition of thermodynamic equilibrium, for which we require that the temperature should be the same everywhere and that we should always find the same value of the temperature, no matter how we measure it. Nothing like this is required for what we call thermal equilibrium. **For thermal equilibrium we require only that the temperature should not change in time.**

So far we have not yet specified by which means the energy is transported. The condition of thermal equilibrium does not specify this. We can have different transport mechanisms at different layers. We know that at the surface the transport must be by radiation. If in the deeper layers the energy transport is also due to radiation, i.e., **if the heat flux F is radiative heat flux, then we talk about *radiative equilibrium*.** As we said, this does not have to be the case. We may have some energy transport by mass motion – the so-called convective energy transport – or we might have energy transport by heat conduction. If all the heat transport were by convection, which, strictly speaking, never happens, we might talk about convective equilibrium. Also, if all the energy transport were due to heat conduction, we might talk about conductive equilibrium. The latter two

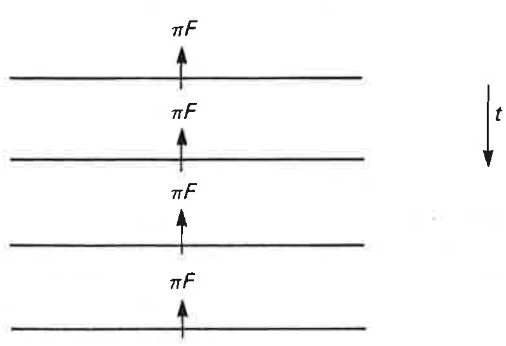

Fig. 6.5. The same amount of energy must be transported through each horizontal layer if the temperature is to remain constant in time.

cases are very rare in stars, so these concepts are hardly ever used. The scheme in Fig. 6.6 demonstrates the situation.

Generally, in the stellar photospheres we find that the condition of radiative equilibrium is a very good approximation. In the solar photosphere we see granulation showing that there is also energy transport by mass motion, but the convective flux is of the order of 1% of the total flux. If there is a temperature gradient there is also some conductive flux, but again, this is much smaller than both the radiative energy flux and the convective flux. When we know how large the temperature gradient is in the solar photosphere and in other stellar atmospheres, we will come back to the discussion of this question. In the following chapters we will only be concerned with radiative equilibrium.

For radiative equilibrium we now consider the energy flux $F = F_r$ to be radiative flux only. Equation (6.9) does not, of course, imply that the same photons keep flowing across the different horizontal planes. There is much absorption and re-emission, but the surplus of radiation in the outward direction, i.e., the net flux $F(\tau)$, has to remain constant.

In order to see what this means for the source function, we have to start from the transfer equation (5.3) which we integrate over the whole solid angle and obtain

$$\int \cos \vartheta \frac{dI_\lambda}{d\tau_\lambda} (\tau_\lambda, \vartheta) \, d\omega = + \int I_\lambda(\tau_\lambda, \vartheta) \, d\omega - \int S_\lambda(\tau_\lambda) \, d\omega. \qquad 6.10$$

Thermodynamic equilibrium: nothing changes in time *and* space

Thermal equilibrium: no temperature changes in time
requires div $F = 0$
with $F = F_{\text{radiative}} + F_{\text{convective}} + F_{\text{conductive}}$

Special cases of thermal equilibrium

Radiative equilibrium	*Convective equilibrium*	*Conductive equilibrium*
$F = F_{\text{radiative}}$ div $F_{\text{radiative}} = 0$ $(F_{\text{convective}} = 0$ $F_{\text{conductive}} = 0)$	$F = F_{\text{convective}}$ div $F_{\text{convective}} = 0$ $(F_{\text{radiative}} = 0$ $F_{\text{conductive}} = 0)$	$F = F_{\text{conductive}}$ div $F_{\text{conductive}} = 0$ $(F_{\text{radiative}} = 0$ $F_{\text{convective}} = 0)$
$F_{\text{radiative}} \gg F_{\text{convective}} + F_{\text{conductive}}$	$F_{\text{convective}} \gg F_{\text{radiative}} + F_{\text{conductive}}$	$F_{\text{conductive}} \gg F_{\text{radiative}} + F_{\text{convective}}$
in the high photospheric layers, and in major fractions of stellar interiors	in the upper layers of the hydrogen convection zone	in the upper parts of the transition regions between stellar chromospheres and coronae. Also in interiors of white dwarfs.

Fig. 6.6 explains the concepts of the different equilibria.

Interchanging differentiation and integration on the left-hand side yields

$$\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}(\pi F_\lambda(\tau_\lambda)) = \int(-S_\lambda(\tau_\lambda) + I_\lambda(\tau_\lambda, \vartheta))\,\mathrm{d}\omega. \qquad 6.11$$

Integrating the right-hand side and dividing by 4π gives

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}F_\lambda(\tau_\lambda) = -S_\lambda(\tau_\lambda) + J_\lambda(\tau_\lambda), \qquad 6.12$$

since S_λ can be assumed to be isotropic and $J_\lambda = \int I_\lambda\,\mathrm{d}\omega/4\pi$, according to (5.32).

If we now again assume κ_λ to be independent of λ, i.e., we assume a *grey atmosphere*, we can then integrate over λ and obtain

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\tau}\int_0^\infty F_\lambda(\tau)\,\mathrm{d}\lambda = -\int_0^\infty S_\lambda(\tau)\,\mathrm{d}\lambda + \int_0^\infty J_\lambda(\tau)\,\mathrm{d}\lambda$$

or 6.13

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}\tau}F(\tau) = -S(\tau) + J(\tau) = 0,$$

according to equation (6.9). Equation (6.13) tells us that

$$S(\tau) = J(\tau). \qquad 6.14$$

In a grey atmosphere, the source function must be equal to the mean intensity J.

If the atmosphere is not grey, it is better to take κ_λ to the right-hand side of equation (6.12). Integration over λ and division by 4π then yields

$$\frac{1}{4}\frac{\mathrm{d}F(\tau_\lambda)}{\mathrm{d}t} = \int_0^\infty(\kappa_\lambda S_\lambda - \kappa_\lambda I_\lambda)\,\mathrm{d}\lambda = 0, \qquad 6.15$$

or

$$\int_0^\infty(\kappa_\lambda S_\lambda - \kappa_\lambda J_\lambda)\,\mathrm{d}\lambda = 0, \qquad 6.16$$

which must always hold if (6.19) holds for the radiative flux, i.e., if we have radiative equilibrium.

The two conditions $\mathrm{d}F/\mathrm{d}t = 0$ and $\int_0^\infty \kappa_\lambda S_\lambda\,\mathrm{d}\lambda = \int_0^\infty \kappa_\lambda J_\lambda\,\mathrm{d}\lambda$ are equivalent.

Equation (6.16) can easily be understood: since $\int_0^\infty \kappa_\lambda S_\lambda\,\mathrm{d}\lambda$ describes the total amount of energy emitted and $\int_0^\infty \kappa_\lambda J_\lambda\,\mathrm{d}\lambda$ describes the total amount of energy absorbed per unit volume, equation (6.16) only says that the amount of energy absorbed must equal the amount of energy re-emitted if no heating or cooling is taking place.

We can understand the functioning of radiative equilibrium by comparing the emitted radiation with water being emitted from one fountain and thrown into the next, where it is 'absorbed' and mixed with the other water and new water is re-emitted and thrown into the next one, etc. Fig. 6.7 illustrates the situation. If the water level is to remain constant, equal amounts of water have to be 'absorbed' and 're-emitted', but the 'absorbed' water is generally not the same as the re-emitted water.

6.4 The theoretical temperature stratification in a grey atmosphere in radiative equilibrium

6.4.1 *Qualitative discussion*

As we saw earlier, the temperature stratification is determined by the amount of energy which has to be transported through the atmosphere. We know from observation that the amount $\pi F = \sigma T_{\text{eff}}^4$ is leaving the surface per cm^2 each second. This is the amount which has to be transported through every horizontal layer of the atmosphere. The larger the amount of energy transport, the larger the temperature gradient has to be. This means that the temperature gradient is expected to grow with increasing T_{eff}^4. We also saw that the more difficult the energy transport, the steeper the temperature gradient has to be. For the photons the obstacles are the

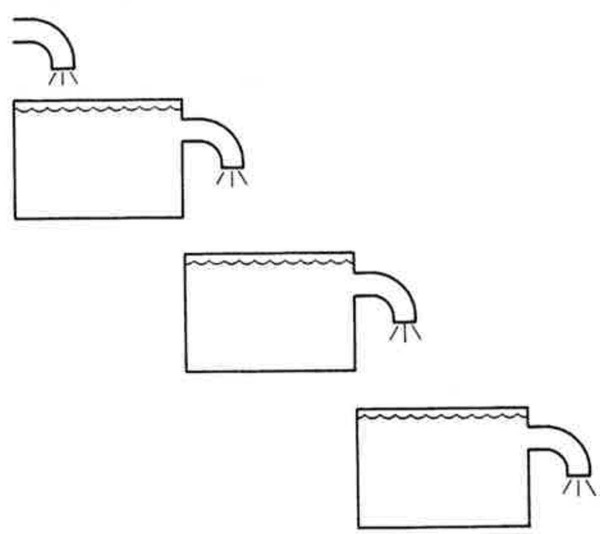

Fig. 6.7 illustrates what is happening in radiative equilibrium. We compare the flow of photons through different horizontal layers with the flow of water through a series of fountains. The water (photons) is emitted from one fountain and flows into the next one. Here it is 'absorbed' and mixed with the rest of the water. A fraction of the mixed water is 're-emitted', i.e., it flows into the next fountain, is 'absorbed' there, mixed with the water (photons) in this fountain and another fraction is 'emitted' again and so on. If the water level (energy level) in all the fountains (horizontal layers of the photosphere) is required to remain constant, then the same amount of water (photons) has to flow out of each faucet into the next fountain in each unit of time.

atoms which keep absorbing them and emitting them equally in all directions. We therefore expect a steeper temperature gradient if the absorption coefficient κ_λ is very large. dT/dt is expected to grow with increasing κ. In fact, without any calculations we can estimate how fast the source function must grow with depth if the radiative flux is to remain constant. We can see this best from a series of drawings (see Fig. 6.8). Again, we compare the photosphere with a number of fountains. We know that on average the photons are absorbed after they have travelled a distance with an optical depth $\Delta\tau_s = 1$ in the direction of propagation. On average this corresponds to a vertical $\Delta\tau = \frac{2}{3}$, when averaged over all directions. We therefore give the fountains a size corresponding to an optical depth $\Delta\tau = \frac{2}{3}$ (see Fig. 6.8(a)). At the surface the flux πF is flowing out. This amount of energy flowing into a half-sphere is indicated by *one* arrow. At the bottom of this layer the same amount – one arrow – has to be the *surplus* of energy flowing up as compared to the energy flowing down. The source function is determined by the photons emitted by the atoms. The atoms have no incentives to emit the photons preferentially into one direction. The source function is therefore expected to be isotropic. Since at the top just one arrow is flowing out, the emission, i.e., the source function, in the top panel must be one arrow in each direction. The source function in the top panel is completely determined by the outflow at the surface. So one arrow is flowing downward from the top fountains into the next deeper layer of fountains. Since the surplus flux upward has to be one arrow, there must be an upward flow of two arrows from the next deeper layer. Again, the source function is isotropic. Therefore, in the next deeper panel there must be an emission of two arrows in all directions. This means the source function must be twice as large as in the top panel. Following this construction to the next deeper layer of fountains, we see easily that for each absorption and re-emission process the emission into a half-sphere must increase by the amount πF leaving the surface into a half-sphere. In Fig. 6.8(b) we consider the energy emitted into half-spheres and absorbed from half-spheres. The energy going into the half-sphere at the surface is given by $\pi F = \sigma T_{\text{eff}}^4$. So for each absorption and re-emission process, i.e., over $\Delta\tau = \frac{2}{3}$, the emission into the half-sphere, which is $2\pi S$, must increase by πF. This means

$$\frac{\Delta(2\pi S)}{\Delta\tau} = \frac{\pi F}{2/3} = \tfrac{3}{2}\pi F \qquad \text{and} \qquad \frac{\Delta S}{\Delta\tau} = \tfrac{3}{4}F. \qquad 6.17$$

Before we start the mathematical derivation, let us look at Fig. 6.8(a) once more and check what the relation is between J, the mean intensity, and S, the source function. The total emission is given by $4\pi S$, by all the outgoing arrows. The total absorption is given by all the ingoing arrows, which correspond to $4\pi J$. We see that, while we have an anisotropy in the radiation field because we always have more upward arrows than downward, we still have the same number of arrows leaving each cell as we have arrows coming in, telling us that, in spite of the anisotropy in the radiation field (determining F), we still have

$$S = J. \qquad 6.18$$

6.4.2 *Mathematical derivation*

Let us now see how the depth dependence of the source function can also be obtained from the transfer equation. We multiply equation

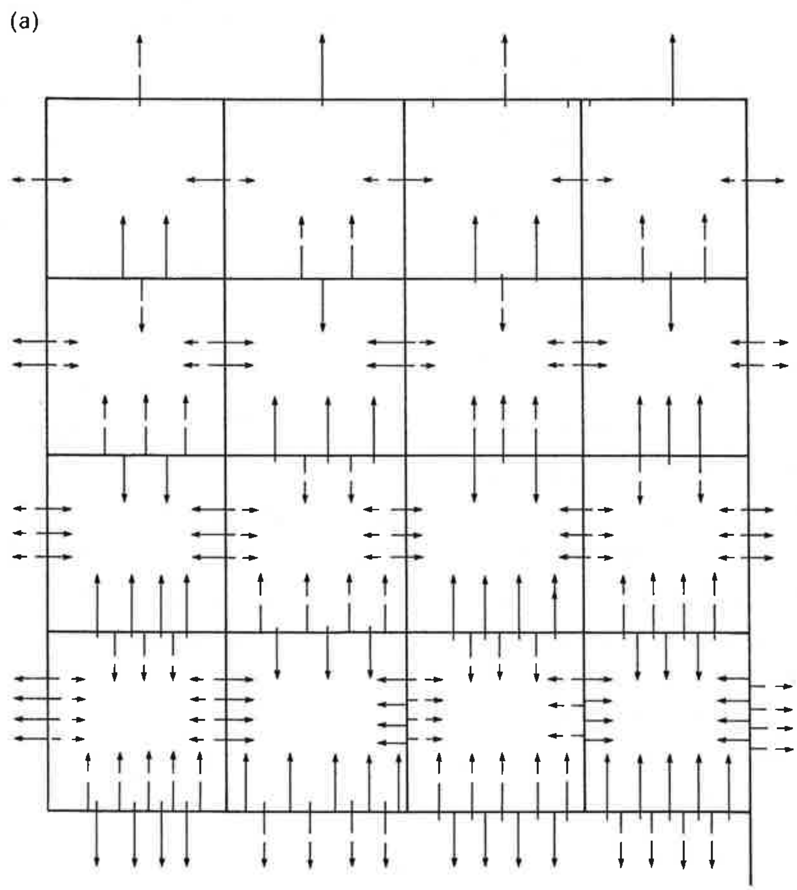

Fig. 6.8. (a) illustrates the situation in an atmosphere with radiative equilibrium. The source function must increase with depth if F has to remain constant. (b) Here we plot the emissions into half-spheres for $\vartheta < \frac{1}{2}\pi$ and $\vartheta > \frac{1}{2}\pi$. The integrals over half-spheres have to grow by πF for each step $\Delta\tau = \frac{2}{3}$.

(5.3) by $\cos\vartheta$ and integrate over all solid angles, ω, to obtain

$$4\pi\frac{\mathrm{d}}{\mathrm{d}\tau_\lambda}K_\lambda=\frac{d}{d\tau_\lambda}\int_{\omega=4\pi}\cos^2\vartheta\, I_\lambda(\tau_\lambda)\,\mathrm{d}\omega$$
$$=\int_{\omega=4\pi}\cos\vartheta\, I_\lambda(\tau_\lambda)\,\mathrm{d}\omega-\int_{\omega=4\pi}\cos\vartheta\, S_\lambda(\tau_\lambda)\,\mathrm{d}\omega. \qquad 6.19$$

The left-hand integral is generally abbreviated by $K_\lambda(\tau_\lambda)$. The last integral on the right-hand side is zero, since $S_\lambda(\tau)$ can be taken to be isotropic. The first integral on the right-hand side equals πF_λ. Division by 4π gives

$$\frac{\mathrm{d}K_\lambda(\tau_\lambda)}{\mathrm{d}\tau_\lambda}=\tfrac{1}{4}F_\lambda(\tau_\lambda). \qquad 6.20$$

We know that the temperature gradient is finally determined by the total energy transport through the atmosphere, i.e., by πF. For radiative

(b)

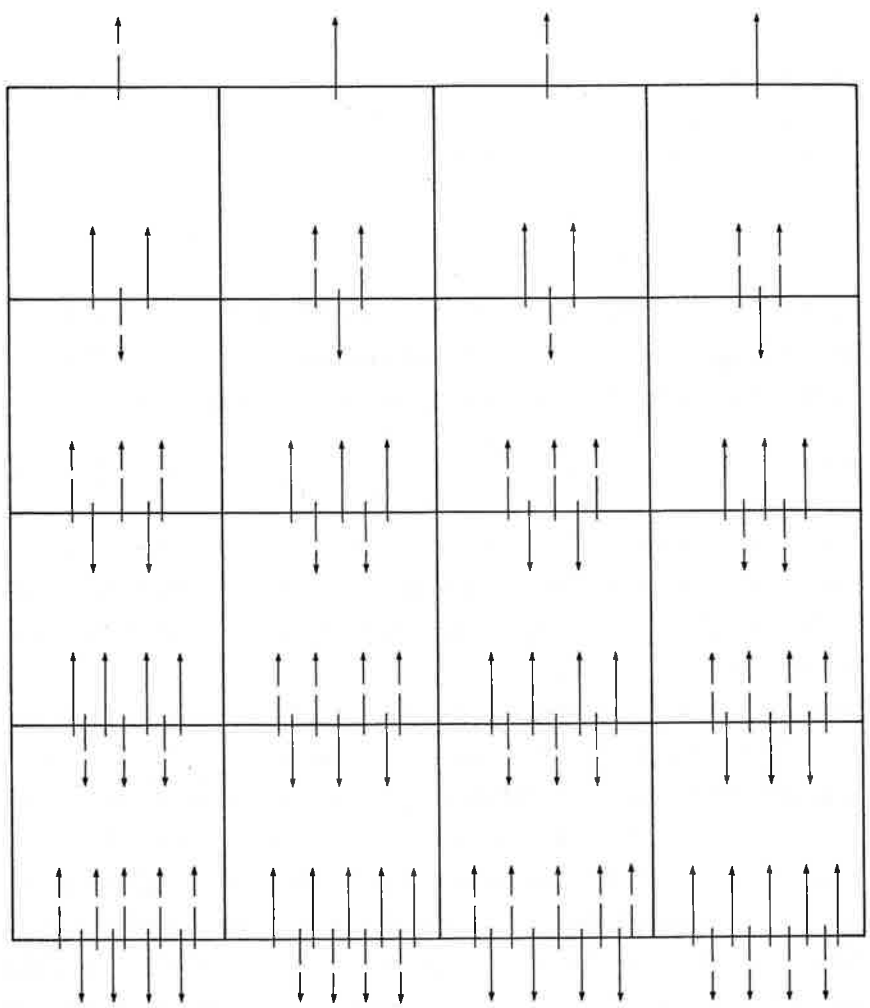

Fig. 6.8(b)

equilibrium, we therefore have to introduce the equation for radiative equilibrium, namely $dF/dt = 0$, into our equations. Since $F = \int_0^\infty F_\lambda \, d\lambda$, we have to integrate the equations over wavelengths λ. However, in equation (6.20) κ_λ occurs in the denominator. If we want to integrate over λ we have to specify the wavelength dependence of κ_λ. We have not yet discussed this, except for the empirical numerical determination for the sun; therefore, we will make the simplest specification; namely, assume that κ is independent of wavelength, which means we now make the approximation of a *grey atmosphere*. Even if the real stellar atmospheres are not grey, we may hope that our results are still approximately correct if we determine an appropriate average value for κ_λ.

If we are dealing with a grey atmosphere we can integrate equation (6.20) over λ and obtain

$$\frac{dK}{d\tau} = \tfrac{1}{4}F, \qquad \text{where } K = \int_0^\infty K_\lambda \, d\lambda. \qquad 6.21$$

This introduces a new unknown function $K(\tau)$. In order to find an equation for $dF/d\tau$ we now differentiate with respect to τ, which, after integrating the transfer equation over λ, yields

$$\frac{d^2K(\tau)}{d\tau^2} = \frac{1}{4}\frac{dF(\tau)}{d\tau} = J(\tau) - S(\tau) = 0, \qquad 6.22$$

because we are dealing with radiative equilibrium, for which $dF/d\tau = 0$. We then find again that for a grey atmosphere $S(\tau) = J(\tau)$. This is our first important result. Integration of the equation with respect to τ gives

$$K = c_1\tau + c_2, \qquad \text{where } \frac{dK}{d\tau} = c_1 = \tfrac{1}{4}\mathrm{F} \text{ such that } K = \tfrac{1}{4}F\tau + c_2. \qquad 6.23$$

With the new unknown function $K(\tau)$ we now have for a given F two equations – (6.22) and (6.23) – for the determination of the three unknowns, K, J, S. We need an additional relation between two of these variables in order to determine all three.

We have seen that for the determination of the flux the anisotropy in the radiation field $I_\lambda(\vartheta)$ is very important because in the flux integral the inward-going intensities are subtracted from the outward-going ones, due to the factor $\cos\vartheta$. For K, on the other hand, a small anisotropy is unimportant because the intensities are multiplied by the factor $\cos^2\vartheta$, which does not change sign for inward and outward radiation. In order to evaluate K, we can therefore approximate the radiation field by an isotropic radiation field of the mean intensity J. From the definition of K_λ we obtain

$$4\pi K_\lambda(\tau_\lambda) = \int_{\omega=4\pi} I_\lambda(\tau_\lambda, \vartheta)\cos^2\vartheta \, d\omega$$

$$= J_\lambda(\tau_\lambda)\int_{\omega=4\pi} \cos^2\vartheta \, d\omega = \frac{4\pi}{3} J_\lambda(\tau_\lambda), \qquad 6.24$$

or, after division by 4π,

$$K_\lambda = \tfrac{1}{3}J_\lambda \quad \text{and} \quad K = \tfrac{1}{3}J, \qquad 6.25$$

when we integrate over λ. This approximation for the K function is called the *Eddington approximation.* It has received wide application and is therefore very important.

Inserting the relation (6.25) into equation (6.21) we find

$$\frac{dK(\tau)}{d\tau} = \frac{1}{3}\frac{dJ(\tau)}{d\tau} = \tfrac{1}{4}F(\tau) = \text{const.} \qquad 6.26$$

From (6.23) we then derive

$$\frac{dK(\tau)}{d\tau} = c_1 = \tfrac{1}{4}F \quad \text{or} \quad \frac{dJ(\tau)}{d(\tau)} = 3\frac{dK(\tau)}{d\tau} = \tfrac{3}{4}F, \qquad 6.27$$

which yields

$$J(\tau) = \tfrac{3}{4}F\tau + \text{const.} = S(\tau). \qquad 6.28$$

Instead of equation (6.17), which we derived from Fig. 6.8(b), we now find again that

$$\frac{dS}{d\tau} = \tfrac{3}{4}F\tau \quad \text{and} \quad S(\tau) = \tfrac{3}{4}F\tau + c_2, \qquad 6.29$$

as we derived from Fig. 6.8(b) (see equation 6.17). *From the condition of radiative equilibrium we thus find the law for the depth dependence of the source function.*

If we want to derive the depth dependence of the temperature from this, we have to know what the relation is between the source function and the temperature. The condition of radiative equilibrium does not tell us this directly. Therefore, we now make the assumption of LTE, which means we now assume that the source function is given by the Planck function. We want to emphasize that the temperature which we obtain as a function of depth depends on this assumption. The depth dependence of the source function is independent of this assumption. With

$$S(\tau) = B(\tau) = \frac{\sigma}{\pi}T^4(\tau), \qquad 6.30$$

we then find

$$\sigma T^4(\tau) = \tfrac{3}{4}\pi F(\tau + \text{const.}) = \pi S(\tau), \qquad 6.31$$

and

$$\sigma T^4(\tau) = \tfrac{3}{4}T_{\text{eff}}^4(\tau + \text{const.}) \quad \text{since } \pi F = \sigma T_{\text{eff}}^4. \qquad 6.32$$

The source function must increase linearly with the optical depth τ and, as we saw, the gradient must be proportional to the flux.

The constant c_2 in equation (6.23) is still undetermined. Instead of this constant we can determine the constant in equation (6.32). These are integration constants which have to be determined from the boundary conditions. Our boundary condition here simply states that there is no flux going into the star. We must have

$$I(0, \vartheta) = I^- = 0 \qquad \text{for } \tfrac{1}{2}\pi < \vartheta \leqslant \pi. \tag{6.33}$$

In order to be able to calculate K and J at the surface of the star we make the simplifying assumption that the outward-going intensity does not depend on the angle ϑ. This means we assume that

$$I(0, \vartheta) = I^+ = \text{const.} \qquad \text{for } 0 < \vartheta \leqslant \tfrac{1}{2}\pi. \tag{6.34}$$

With this the integration over the whole solid angle reduces to an integration over the half-sphere which gives

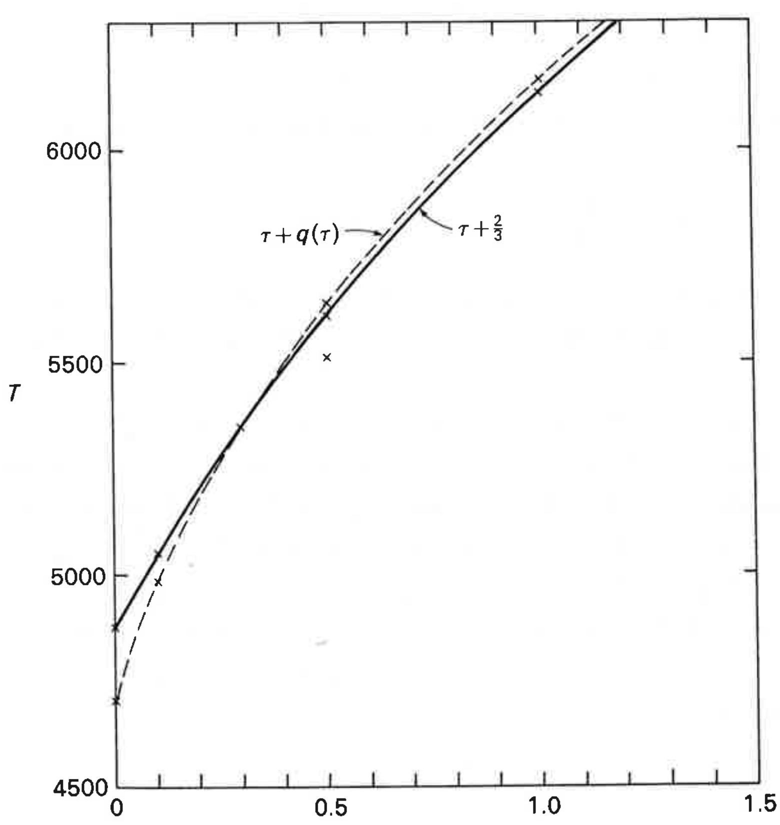

Fig. 6.9. The exact temperature stratification of a grey atmosphere (---) is compared with the simple approximation (——) given by equation (6.38). The exact solution gives somewhat lower surface temperatures and slightly higher temperatures in deeper layers.

$$J(0) = \tfrac{1}{2}I^+ \qquad \text{and} \qquad F = I^+ \qquad \text{or} \qquad J(0) = \tfrac{1}{2}F, \tag{6.35}$$

which means

$$J(0) = B(0) = \frac{\sigma}{\pi}\, T^4(0) \qquad \text{or} \qquad T^4(0) = \tfrac{1}{2}T^4_{\text{eff}}. \tag{6.36}$$

From equation (6.13) we derive for $\tau = 0$

$$T^4(0) = \tfrac{3}{4}T^4_{\text{eff}} = \text{const.} = \tfrac{1}{2}T^4_{\text{eff}}, \tag{6.37}$$

which requires const. $= \frac{2}{3}$.

We finally derive for the grey atmosphere with the Eddington approximation

$$T^4(\tau) = \tfrac{3}{4}T^4_{\text{eff}}(\tau + \tfrac{2}{3}). \tag{6.38}$$

For the optical depth $\tau = \frac{2}{3}$ we then find $T^4(\tau = \frac{2}{3}) = T^4_{\text{eff}}$ or $T(\tau = \frac{2}{3}) = T_{\text{eff}}$, as we derived previously for a grey atmosphere.

Without making the Eddington approximation and with the accurate boundary condition, i.e., considering the true angular dependence of $I(0, \vartheta)$, one obtains

$$T^4(\tau) = \tfrac{3}{4}T^4_{\text{eff}}(\tau + q(\tau)), \tag{6.39}$$

where $q(\tau)$ is a function which varies slowly with τ, with $q(0) = 0.577$ and $q(\infty) = 0.7101$. So $q(\tau) = \frac{2}{3}$ is quite good as an approximation. In Fig. 6.9 we compare the temperature stratifications obtained with (6.38) and (6.39).

How good an approximation is the grey atmosphere? In order to judge this, we have to discuss the frequency dependence of the absorption coefficients.