GLOSSARY
(in order of appearance and usefulness)

<u>Tensor</u>: A tensor is basically a multi-dimensional matrix. We usually depict a tensor with indices to denote its different dimensions. For example, if B = (1, 2, 3, 4) and C = (0, 1, 0, 2), then

$$A_{\mu\nu} = B_\mu C_\nu$$

defines A as the two-dimensional tensor

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 4 \\ 0 & 3 & 0 & 6 \\ 0 & 4 & 0 & 8 \end{pmatrix}.$$

Likewise, if $A_{\mu\nu\lambda} = B_\mu B_\nu C_\lambda$, then we'd have $A_{134} = B_1 B_3 C_4 = 1 \times 3 \times 2 = 6$. A one-dimensional tensor is a vector and a zero-dimensional tensor is a scalar. In this paper, I think we only deal with two-dimensional tensors, so it's safe to think of them as matrices.

<u>Metric Tensor</u>: The metric tensor $g_{\mu\nu}$ tells you about the geometry of space and time. In particular, the metric defines an inner product that is the equivalent of the dot product in a given space (the dot product is only valid in Euclidean space). This inner product of V with W is defined as the product $V^{\mathrm{T}}(g_{\mu\nu})W$. For example, in Euclidian space, the metric is simply the identity matrix, so we retrieve the familiar definition of the dot product. In special relativity, the metric is given by the Minkowski tensor

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

which leads to the well-known invariant $ds^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2$ when we take the inner product of the vector ($dx, dy, dz, c\ dt$) with itself.

<u>Raised and Lowered Indices:</u>

The inner product is often written as $V_\mu W^\mu$ or $g_{\mu\nu} V^\nu W^\mu$. There are two new things going on in this definition. First of all, when the same index label appears twice in a given term, we're supposed to sum over that index (this is Einstein summation notation). Thus,

$$B_\mu C_\mu = B_1 C_1 + B_2 C_2 + B_3 C_3 + B_4 C_4 = 0 + 2 + 0 + 8 = 10 .$$

We also see that the metric lowers the index of V. In general,

$$g_{\mu\nu} A_{abcd\ldots}{}^{\alpha\beta\gamma\ldots\nu\ldots} = A_{abc\ldots\mu\ldots}{}^{\alpha\beta\gamma\delta\ldots}.$$

A is any tensor. The inverse of $g_{\mu\nu}$, $g^{\mu\nu}$, raises indices. We can think of raising and lowering as taking the transpose in our space. In Euclidean space, to take the dot product of V with W, we transpose V and use matrix multiplication. In the product $g_{\mu\nu} V^{\nu} W^{\mu}$, we first use $g_{\mu\nu}$ to take the transpose of V, and then multiply V-transpose and W. In the equivalent product $V_{\mu} W^{\mu}$, we've already collapsed the sum over ν and simply depict the transposed-ness of V with a lowered index.

<u>Commas in Indices:</u> denote a partial derivative with respect to the coordinate that follows the comma.

<u>Functional</u>: Simply a scalar function that takes a vector as its argument.

<u>Action</u>: By now, you've probably heard of the principle of "least time." Well, there's also the principle of least action. The action is the integral of the Lagrangian over a path:

$$S = \int_{path} L dt \, .$$

Of course, with a change of variables, time can be replaced by distance along the path. The Lagrangian is a measure of the energy of the system, and is defined by

L = Kinetic energy – Potential energy

L can be a function of both time and space. By minimizing the action with respect to position, we can determine equations of motion for the objects in our system. You'll do lots of this in 111.

<u>Einstein-Hilbert Action</u>:

$$S = \int \left(\frac{c^4}{16\pi G} R \sqrt{-g} \right) d^4 x$$

c and G are the speed of light and the gravitational constant. g is the trace of the metric tensor, and R is the *scalar curvature*, which is determined by the metric and coordinate system. We thus see that the Einstein-Hilbert action depends only on the choice of coordinates and the metric tensor $g_{\mu\nu}$. Minimizing this action produces Einstein's field equations.

<u>Friedmann-Robertson-Walker Metric:</u> The FRW metric is a solution to the Einstein field equations. In Cartesian coordinates, it's given by

$$g_{\mu\nu} = \begin{pmatrix} a(t)^2 & 0 & 0 & 0 \\ 0 & a(t)^2 & 0 & 0 \\ 0 & 0 & a(t)^2 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$a(t)$ is known as the scale factor, as it scales the spatial coordinates with respect to the time coordinate.

SUMMARY

Dark matter explains observations such as galaxy rotation curves, gravitational lensing, and large scale structure that can't be understood with only baryonic matter and our current laws of gravity. However, dark matter is problematic – or rather, dark energy is. We don't have any good theories that explain it, and the theories that do offer an explanation tend to predict more dark energy than we actually observe. If we assume that our universe can be explained with our current theories of gravity, then we must allow both dark matter and dark energy, the so-called "dark sector."

To get rid of the dark sector, Milgrom proposed a theory of modified Newtonian dynamics, or MOND. MOND says that, for low accelerations, Newtonian gravity no longer holds. Bekenstein expanded on Milgrom's theory and came up with the math required for a full relativistic model. Basically, Bekenstein redefined the metric, which is usually a tensor field, to also include a tensor field and a scalar field. His theory is called TeVeS, for the tensor, vector, and scalar nature of the fields.

In this paper, the authors apply the TeVeS model to a specific problem: the growth of density perturbations in the early universe. As seen in Figure 1, without dark matter, the Newtonian-gravity density fluctuation power spectrum has a much lower amplitude than the observed spectrum (I believe that the x-axis scales as 1/fluctuation size and the y-axis is a measure of how much clumping there is at a given scale). A large amplitude is required to form large-scale structure, so such structure cannot form with only baryonic matter according to Newtonian gravity. The authors note that numerical simulations of TeVeS have already been shown to increase the growth of perturbations in order to produce a larger amplitude, and so they'd like to analytically determine the root of the growth. Note that, in Figure 1, we also see that the shape of the baryonic power spectrum is different (more wiggly) than the observed spectrum, but the authors do not address this issue in the paper.

Next up is the mathematical model. Instead of taking you through it step by step, I'll compare the results of general relativity with those of TeVeS:

	general relativity	TeVeS
metric	$g_{\mu\nu}$ is governed by the Einstein-Hilbert action	$g_{\mu\nu} \equiv e^{-2\phi}(\tilde{g}_{\mu\nu} + A_\mu A_\nu) - e^{2\phi} A_\mu A_\nu$, where $\tilde{g}_{\mu\nu}$ is governed by the Einstein-Hilbert action, *A* is a vector field governed by the vector action defined below, and ϕ is a scalar field governed by the scalar action defined below. Indices are raised and lowered by $\tilde{g}_{\mu\nu}$.
actions	Einstein-Hilbert action: $S = \int \left(\frac{c^4}{16\pi G} R\sqrt{-g} \right) d^4x$	Einstein Hilbert action for the tensor field: $S = \int \left(\frac{c^4}{16\pi G} \tilde{R}\sqrt{-\tilde{g}} \right) d^4x$ Scalar field action: $S_S = \frac{-c^4}{16\pi G} \int d^4x \sqrt{-\tilde{g}} \left[\mu\left(\tilde{g}^{\mu\nu} - A^\mu A^\nu\right)\phi_{,\mu}\phi_\nu + V \right]$ Vector field action: $S_V = \frac{-c^4}{32\pi G} \int d^4x \sqrt{-\tilde{g}} \left[KF^{\alpha\beta}F_{\alpha\beta} - 2\lambda\left(A^2 + 1\right) \right]$ *F* is defined by *A*. μ is a scalar field. λ is a scalar fixed by the action. The potential *V* (Eq. 3) introduces two constant free parameters, μ_0 and ℓ_B. Additionally, there's a *K* in the vector field action, which maybe relates to Killing vectors? Or perhaps *K* is actually the third free parameter, K_B, with the subscript left off.
time evolution of the scale factor	Friedmann equation: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2}$	Same, but *G* becomes $G\frac{e^{-4\phi}}{\left[1 + d\phi / d\ln a\right]^2}$, and there is an additional term for the energy density of the scalar field.

We thus see that the TeVeS model is quite a bit more complex than the standard GR model. In GR, it is typical to guess at the form of the action and consider it true if it predicts observations correctly. However, the forms of the TeVeS actions are so complex that it is difficult to understand how Bekenstein came up with them.

Next, in order to investigate growth of density fluctuations, the authors perturb the system. For a normal FRW metric, the perturbation would consist only of a perturbation of the matter and radiation densities. For the TeVeS model, the scalar and vector fields also enter into the calculation. In the third panel of Figure 3, we see that, for small values of the free parameter K_B, there is greater growth of perturbations than for the standard model. This enhanced growth is due to perturbations in the vector field, and without vector perturbations, growth is not enhanced.