Astronomy 1 – Introductory Astronomy

Spring 2014

Error/Uncertainty Analysis

When we talk about "error" in science, we are not using the term in the everyday sense of a "mistake." Instead, "error" is a way of quantifying how confident we are that a measurement is the "true" or "correct" value. The term "uncertainty" is often used interchangeably (and gives a more accurate description of what we're describing here), but the phrase "error analysis" is in common usage. This document discusses estimation of these uncertainties, how those uncertainty values *propagate* (i.e. translate into uncertainties on values calculated from measured quantities), and how the value of the uncertainty gives guidance on the number of significant digits to write for a value.

First, every quantity you measure or estimate has some uncertainty associated with it, so in every lab you should give an estimate of the **experimental uncertainty** associated with your actual measurements. In arriving at this estimate, you should consider factors such as:

- *the precision of your instrument (e.g. is your ruler marked off in millimeters or centimeters?),*
- *your own proficiency with the instruments (if you are using a centimeter ruler, can you confidently read to the nearest centimeter or the nearest half-centimeter?),* and
- *any variation of the quantity you are trying to measure (how well can you use a ruler to measure the location of a moving ball at a given time?)*

Second, you will almost always use your measured values to calculate some other values of interest. Thus, these calculated values will have some uncertainty as well, so you must **propagate** these uncertainty estimates in order to calculate the uncertainty estimates for all the important quantities calculated from your actual measurements. In other words, how do the experimental uncertainty estimates translate into an uncertainty in the final answer?

Determining how uncertainty estimates propagate is not difficult. Imagine that you have measured two distances: a distance x_1 equal to 348 cm with an error estimate of 5 cm, and a distance x_2 equal to 821 cm with an error estimate of 9 cm. If you calculate the total distance or the difference in distances, what is the uncertainty estimate for these two quantities? Your first thought might be to add the two uncertainty estimates together. This is a good first approximation, but we can do better. We expect the error in the new distance to be more than that of either x_1 or x_2 individually, but it is likely to be less than the sum of the two errors.[1] An easy (and correct) way to get an uncertainty estimate between these two extremes is to add the uncertainties **in quadrature**. This means to add the **squares** of the uncertainties together, then take the square root (like the Pythagorean theorem in geometry). In our example:

$$\text{combined uncertainty} = \sqrt{5^2 + 9^2} = \sqrt{106} \approx 10.3 \text{ cm}$$

Since the distances were measured only to the nearest centimeter, and we see that in fact our answer is uncertain by a full 10 cm, it does not make sense to report an uncertainty with a precision of 0.1 cm. So the answers are:

$x_2 + x_1 =$ 1,169 cm with an error estimate of 10 cm,
$x_2 - x_1 =$ 473 cm with an error estimate of 10 cm.

A convenient way of writing these results is

$$x_2 + x_1 = (1169 \pm 10) \text{ cm} \qquad \text{and} \qquad x_2 - x_1 = (473 \pm 10) \text{ cm}.$$

Another example of this compact way of writing uncertainties for numbers written in scientific notation: if we measure the speed of light to be 3.1×10^8 m/s with an uncertainty of 0.1×10^8 m/s, we generally write this as $(3.1 \pm 0.1) \times 10^8$ m/s.

Now let's imagine that you have measured the time t it took an object to go the total distance $x_1 + x_2$. Let's say t is equal to 3.4 seconds with an uncertainty estimate of 0.2 seconds. If you calculate the velocity of the object, what is the uncertainty estimate for it? Simply adding the uncertainties in quadrature does not work in this case since distance and time are in different units. Instead, we can **add the** uncertainty **estimates**

[1] This is true as long as the uncertainty on each measurement is *random* (equally likely to go in either direction).

as percentages (again in quadrature) in order to get the uncertainty estimate of the product or quotient as a percentage. So in this case,

$$\%(x_1 + x_2) = \frac{10 \text{ cm}}{1169 \text{ cm}} \cdot 100 = 0.86\%,$$

$$\%(t) = \frac{0.2 \text{ s}}{3.4 \text{ s}} \cdot 100 = 5.9\%,$$

$$\%(v) = \sqrt{(0.86\%)^2 + (5.9\%)^2} = 6.0\%.$$

So the velocity v is equal to 343.8 cm/s (1169 cm divided by 3.4 s) with an uncertainty estimate of 6.0% of the velocity, or 20.6 cm/s (6.0% of 343.8 cm/s). Again, the size of the uncertainty gives us guidance on how many significant digits to write down. It makes little sense to keep track of digits representing more than two digits of the error estimate, (i.e. an uncertainty of 20.6 cm/s means we're not sure of the tens digit of the answer, let alone the tenths), so the final result is written

$$v = (344 \pm 21) \text{ cm/s}.$$

At times, you will be using laboratory analysis software to calculate results, and the program will calculate uncertainty values for you. A particularly common example is that a software fitting package will give you an uncertainty estimate on the slope of a best-fit line you fit to some measured values. Feel free to use these values where appropriate and *note in your report what software package was used to obtain them*.

Finally, it's good to keep in mind the distinction between random errors and systematic errors. Random errors are just as likely to be positive as negative (too high or too low), and they arise from incompleteness in data and our finite abilities to measure things. For example, the fact that a political poll doesn't ask every single voter for their preference but only a finite subset will inevitably lead to random errors. But it may also lead to systematic errors if, for example, there is correlation between how likely someone is to be included in the poll and what their position on the candidate or issue is. The classic example in the early era of political polling involves the different likelihoods that a voter will have a telephone depending on their party affiliation.

For another example of the distinction, consider the errors associated with measuring the length of an object with a ruler. If your hand is shaking, random errors will appear in your data. But if the ruler is mislabeled so that each demarcated inch is really just $7/8^{th}$ of an inch, then a systematic error will be introduced (in the sense that all measurements will be too big).

Note that for random errors, repeated measurements (e.g. adding more people to the polling sample) will lead to better and better cancelation of errors and a lower overall error. But the same is, unfortunately, not true for systematic errors.