transition to the white dwarf stage.) All of these stars have multiple periods, simultaneously displaying at least 3, and as many as 125, different frequencies. Astronomers are deciphering the data to obtain a detailed look at the structure of white dwarfs.

16.3 ■ THE PHYSICS OF DEGENERATE MATTER

We now delve below the surface to ask, What can support a white dwarf against the relentless pull of its gravity? It is easy to show (Problem 16.4) that normal gas and radiation pressure are completely inadequate. The answer was discovered in 1926 by the British physicist Sir Ralph Howard Fowler (1889–1944), who applied the new idea of the Pauli exclusion principle (recall Section 5.4) to the electrons within the white dwarf. The qualitative argument that follows elucidates the fundamental physics of the **electron degeneracy pressure** described by Fowler.

The Pauli Exclusion Principle and Electron Degeneracy

Any system—whether an atom of hydrogen, an oven filled with blackbody photons, or a box filled with gas particles—consists of quantum states that are identified by a set of quantum numbers. Just as the oven is filled with standing waves of electromagnetic radiation that are described by three quantum numbers (specifying the number of photons of wavelength λ traveling in the x-, y-, and z-directions), a box of gas particles is filled with standing de Broglie waves that are also described by three quantum numbers (specifying the particle's component of momentum in each of three directions). If the gas particles are fermions (such as electrons or neutrons), then the Pauli exclusion principle allows at most one fermion in each quantum state because no two fermions can have the same set of quantum numbers.

In an everyday gas at standard temperature and pressure, only one of every 10^7 quantum states is occupied by a gas particle, and the limitations imposed by the Pauli exclusion principle become insignificant. Ordinary gas has a *thermal* pressure that is related to its temperature by the ideal gas law. However, as energy is removed from the gas and its temperature falls, an increasingly large fraction of the particles are forced into the lower energy states. If the gas particles are fermions, only one particle is allowed in each state; thus all the particles cannot crowd into the ground state. Instead, as the temperature of the gas is lowered, the fermions will fill up the lowest available unoccupied states, starting with the ground state, and then successively occupy the excited states with the lowest energy. Even in the limit $T \rightarrow 0$ K, the vigorous motion of the fermions in excited states produces a pressure in the fermion gas. At zero temperature, *all* of the lower energy states and *none* of the higher energy states are occupied. Such a fermion gas is said to be completely **degenerate**.

The Fermi Energy

The maximum energy (ε_F) of any electron in a completely degenerate gas at $T = 0$ K is known as the **Fermi energy**; see Fig. 16.5. To determine this limiting energy, imagine a three-dimensional box of length L on each side. Thinking of the electrons as being standing

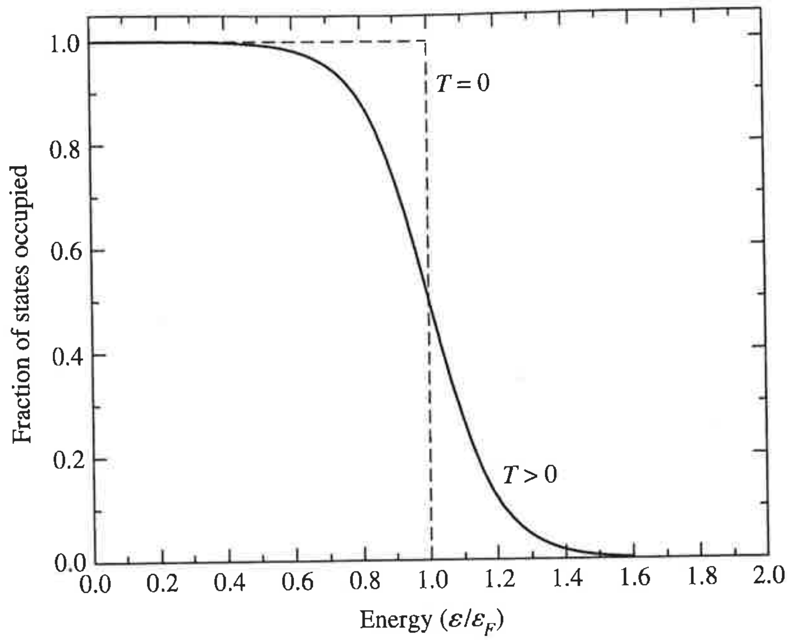

FIGURE 16.5 Fraction of states of energy ε occupied by fermions. For $T = 0$, all fermions have $\varepsilon \leq \varepsilon_F$, but for $T > 0$, some fermions have energies in excess of the Fermi energy.

waves in the box, we note that their wavelengths in each dimension are given by

$$\lambda_x = \frac{2L}{N_x}, \qquad \lambda_y = \frac{2L}{N_y}, \qquad \lambda_z = \frac{2L}{N_z},$$

where N_x, N_y, and N_z are integer quantum numbers associated with each dimension. Recalling that the de Broglie wavelength is related to momentum (Eq. 5.17),

$$p_x = \frac{hN_x}{2L}, \qquad p_y = \frac{hN_y}{2L}, \qquad p_y = \frac{hN_x}{2L}.$$

Now, the total kinetic energy of a particle can be written as

$$\varepsilon = \frac{p^2}{2m},$$

where $p^2 = p_x^2 + p_y^2 + p_z^2$. Thus,

$$\varepsilon = \frac{h^2}{8mL^2}(N_x^2 + N_y^2 + N_z^2) = \frac{h^2 N^2}{8mL^2}, \tag{16.2}$$

where $N^2 \equiv N_x^2 + N_y^2 + N_z^2$, analogous to the "distance" from the origin in "N-space" to the point (N_x, N_y, N_z).

The total number of electrons in the gas corresponds to the total number of unique quantum numbers, N_x, N_y, and N_z times two. The factor of two arises from the fact that electrons are spin $\frac{1}{2}$ particles, so $m_s = \pm 1/2$ implies that two electrons can have the same combination of N_x, N_y, and N_z and still posses a unique set of *four* quantum numbers (including spin). Now, each integer coordinate in N-space (e.g., $N_x = 1, N_y = 3, N_z = 1$) corresponds to the quantum state of two electrons. With a large enough sample of electrons, they can be thought of as occupying each integer coordinate out to a radius of $N = \sqrt{N_x^2 + N_y^2 + N_z^2}$, but only for the positive octant of N-space where $N_x > 0$, $N_y > 0$, and $N_z > 0$. This means that the total number of electrons will be

$$N_e = 2\left(\frac{1}{8}\right)\left(\frac{4}{3}\pi N^3\right).$$

Solving for N yields

$$N = \left(\frac{3N_e}{\pi}\right)^{1/3}.$$

Substituting into Eq. (16.2) and simplifying, we find that the Fermi energy is given by

$$\boxed{\varepsilon_F = \frac{\hbar^2}{2m}\left(3\pi^2 n\right)^{2/3},} \tag{16.3}$$

where m is the mass of the electron and $n \equiv N_e/L^3$ is the number of electrons per unit volume. The average energy per electron at zero temperature is $\frac{3}{5}\varepsilon_F$. (Of course the derivation above applies for any fermion, not just electrons.)

The Condition for Degeneracy

At any temperature above absolute zero, some of the states with an energy less than ε_F will become vacant as fermions use their thermal energy to occupy other, more energetic states. Although the degeneracy will not be precisely complete when $T > 0$ K, the assumption of complete degeneracy is a good approximation at the densities encountered in the interior of a white dwarf. All but the most energetic particles will have an energy less than the Fermi energy. To understand how the degree of degeneracy depends on both the temperature and the density of the white dwarf, we first express the Fermi energy in terms of the density of the electron gas. For full ionization, the number of electrons per unit volume is

$$n_e = \left(\frac{\#\text{ electrons}}{\text{nucleon}}\right)\left(\frac{\#\text{ nucleons}}{\text{volume}}\right) = \left(\frac{Z}{A}\right)\frac{\rho}{m_H}, \tag{16.4}$$

where Z and A are the number of protons and nucleons, respectively, in the white dwarf's nuclei, and m_H is the mass of a hydrogen atom.[9] Thus the Fermi energy is proportional to the 2/3 power of the density,

$$\varepsilon_F = \frac{\hbar^2}{2m_e}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3}. \tag{16.5}$$

[9]The hydrogen mass is adopted as a representative mass of the proton and neutron.

Now compare the Fermi energy with the average thermal energy of an electron, $\frac{3}{2}kT$ (where k is Boltzmann's constant; see Eq. 10.17). In rough terms, if $\frac{3}{2}kT < \varepsilon_F$, then an average electron will be unable to make a transition to an unoccupied state, and the electron gas will be degenerate. That is, for a degenerate gas,

$$\frac{3}{2}kT < \frac{\hbar^2}{2m_e}\left[3\pi^2\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{2/3},$$

or

$$\frac{T}{\rho^{2/3}} < \frac{\hbar^2}{3m_e k}\left[\frac{3\pi^2}{m_H}\left(\frac{Z}{A}\right)\right]^{2/3} = 1261 \text{ K m}^2 \text{ kg}^{-2/3}$$

for $Z/A = 0.5$. Defining

$$\mathcal{D} \equiv 1261 \text{ K m}^2 \text{ kg}^{-2/3},$$

the condition for degeneracy may be written as

$$\boxed{\frac{T}{\rho^{2/3}} < \mathcal{D}.} \tag{16.6}$$

The smaller the value of $T/\rho^{2/3}$, the more degenerate the gas.

Example 16.3.1. How important is electron degeneracy at the centers of the Sun and Sirius B? At the center of the standard solar model (see Table 11.1), $T_c = 1.570 \times 10^7$ K and $\rho_c = 1.527 \times 10^5$ kg m^{-3}. Then

$$\frac{T_c}{\rho_c^{2/3}} = 5500 \text{ K m}^2 \text{ kg}^{-2/3} > \mathcal{D}.$$

In the Sun, electron degeneracy is quite weak and plays a very minor role, supplying only a few tenths of a percent of the central pressure. However, as the Sun continues to evolve, electron degeneracy will become increasingly important (Fig. 16.6). As described in Section 13.2, the Sun will develop a degenerate helium core while on the red giant branch of the H–R diagram, leading eventually to a core helium flash. Later, on the asymptotic giant branch, the progenitor of a carbon–oxygen white dwarf will form in the core to be revealed when the Sun's surface layers are ejected as a planetary nebula.

For Sirius B, the values of the density and central temperature estimated above lead to

$$\frac{T_c}{\rho_c^{2/3}} = 37 \text{ K m}^2 \text{ kg}^{-2/3} \ll \mathcal{D},$$

so complete degeneracy is a valid assumption for Sirius B.

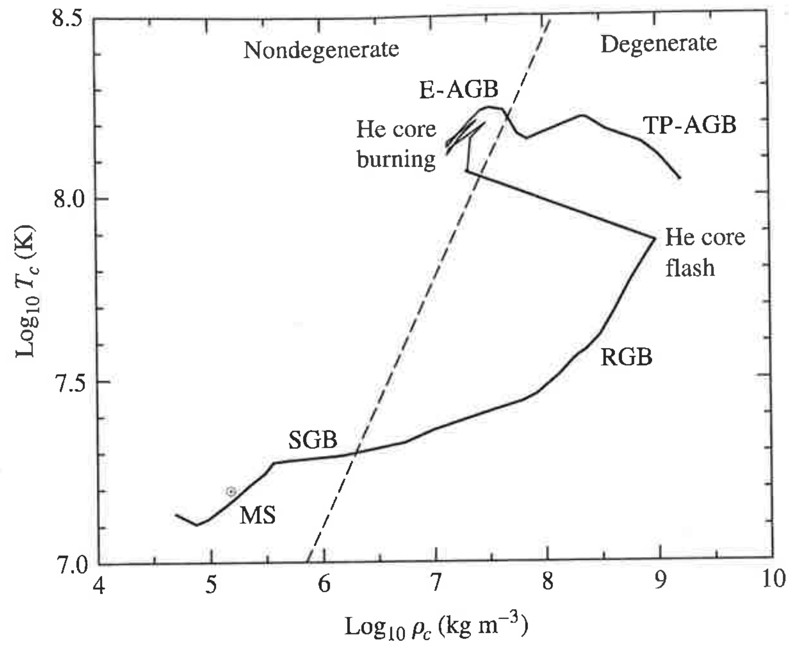

FIGURE 16.6 Degeneracy in the Sun's center as it evolves. (Data from Mazzitelli and D'Antona, *Ap. J.*, *311*, 762, 1986.)

Electron Degeneracy Pressure

We now estimate the electron degeneracy pressure by combining two key ideas of quantum mechanics:

1. The Pauli exclusion principle, which allows at most one electron in each quantum state; and
2. Heisenberg's uncertainty principle in the form of Eq. (5.19),

$$\Delta x\, \Delta p_x \approx \hbar,$$

which requires that an electron confined to a small volume of space have a correspondingly high uncertainty in its momentum. Because the minimum value of the electron's momentum, p_{min}, is approximately Δp, more closely confined electrons will have greater momenta.

When we make the unrealistic assumption that all of the electrons have the same momentum, p, Eq. (10.8) for the pressure integral becomes

$$P \approx \frac{1}{3} n_e p v, \tag{16.7}$$

where n_e is the total electron number density.

In a completely degenerate electron gas, the electrons are packed as tightly as possible, and for a uniform number density of n_e, the separation between neighboring electrons is about $n_e^{-1/3}$. However, to satisfy the Pauli exclusion principle, the electrons must maintain their identities as different particles. That is, the uncertainty in their positions cannot be larger than their physical separation. Identifying $\Delta x \approx n_e^{-1/3}$ for the limiting case of complete degeneracy, we can use Heisenberg's uncertainty relation to estimate the momentum of an electron. In one coordinate direction,

$$p_x \approx \Delta p_x \approx \frac{\hbar}{\Delta x} \approx \hbar n_e^{1/3} \tag{16.8}$$

(see Example 5.4.2). However, in a three-dimensional gas each of the directions is equally likely, implying that

$$p_x^2 = p_y^2 = p_z^2,$$

which is just a statement of the equipartition of energy among all the coordinate directions. Therefore,

$$p^2 = p_x^2 + p_y^2 + p_z^2 = 3p_x^2,$$

or

$$p = \sqrt{3}p_x.$$

Using Eq. (16.4) for the electron number density with full ionization gives

$$p \approx \sqrt{3}\hbar \left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{1/3}.$$

For nonrelativistic electrons, the speed is

$$v = \frac{p}{m_e}$$

$$\approx \frac{\sqrt{3}\hbar}{m_e} n_e^{1/3} \tag{16.9}$$

$$\approx \frac{\sqrt{3}\hbar}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{1/3}. \tag{16.10}$$

Inserting Eqs. (16.4), (16.8), and (16.10) into Eq. (16.7) for the electron degeneracy pressure results in

$$P \approx \frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3}. \tag{16.11}$$

This is roughly a factor of two smaller than the exact expression for the pressure due to a completely degenerate, nonrelativistic electron gas P,

$$P = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e} n_e^{5/3},$$

or

$$P = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3}. \tag{16.12}$$

Using $Z/A = 0.5$ for a carbon–oxygen white dwarf, Eq. (16.12) shows that the electron degeneracy pressure available to support a white dwarf such as Sirius B is about 1.9×10^{22} N m^{-2}, within a factor of two of the estimate of the central pressure made previously (Eq. 16.1). *Electron degeneracy pressure is responsible for maintaining hydrostatic equilibrium in a white dwarf.*

You may have noticed that Eq. (16.12) is the polytropic equation of state, $P = K\rho^{5/3}$, corresponding to $n = 1.5$. This implies that the extensive tools associated with the Lane–Emden equation (Eq. 10.110), developed beginning on page 334, can be used to study these objects. Of course, to understand them in detail requires careful numerical calculations involving the details of the complex equation of state of partially degenerate gases, nonzero temperatures, and changing compositions.

16.4 ■ THE CHANDRASEKHAR LIMIT

The requirement that degenerate electron pressure must support a white dwarf star has profound implications. In 1931, at the age of 21, the Indian physicist Subrahmanyan Chandrasekhar announced his discovery that *there is a maximum mass for white dwarfs*. In this section we will consider the physics that leads to this amazing conclusion.

The Mass–Volume Relation

The relation between the radius, $R_{\rm wd}$, of a white dwarf and its mass, $M_{\rm wd}$, may be found by setting the estimate of the central pressure, Eq. (16.1), equal to the electron degeneracy pressure, Eq. (16.12):

$$\frac{2}{3}\pi G\rho^2 R_{\rm wd}^2 = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left[\left(\frac{Z}{A}\right)\frac{\rho}{m_H}\right]^{5/3}.$$

Using $\rho = M_{\rm wd}/\frac{4}{3}\pi R_{\rm wd}^3$ (assuming constant density), this leads to an estimate of the radius of the white dwarf,

$$R_{\rm wd} \approx \frac{(18\pi)^{2/3}}{10}\frac{\hbar^2}{Gm_e M_{\rm wd}^{1/3}}\left[\left(\frac{Z}{A}\right)\frac{1}{m_H}\right]^{5/3}. \tag{16.13}$$

For a 1 $M_\odot$ carbon–oxygen white dwarf, $R \approx 2.9 \times 10^6$ m, too small by roughly a factor of two but an acceptable estimate. More important is the surprising implication that $M_{\rm wd}R_{\rm wd}^3 =$ constant, or

$$M_{\rm wd}V_{\rm wd} = \text{constant}. \tag{16.14}$$