The metric on the sphere

An alternative derivation of the metric on the sphere starts with the equation for the sphere itself:

$$x^2 + y^2 + z^2 = R^2. \tag{1}$$

If we work in polar coordinates (so that $x^2 + y^2 = r^2$) then the equation for the sphere can be rewritten as

$$r^2 + z^2 = R^2. \tag{2}$$

The Euclidean metric $ds^2 = dx^2 + dy^2 + dz^2$ can be written in polar coordinates as

$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2. \tag{3}$$

We can take the metric in 3D Euclidean space and apply it to the surface of our embedded sphere. To do this Eq. (2) tells us how dr and dz must be related if we can only move on the surface of the sphere: we have that $zdz = -rdr$. Plugging this into the metric in 3D Euclidean space gives us a metric on the surface of the sphere:

$$ds^2 = dr^2 + r^2 d\theta^2 + \frac{r^2}{z^2} dr^2 = \left(1 + \frac{r^2}{z^2}\right) dr^2 + r^2 d\theta^2. \tag{4}$$

We can again use Eq. (2) to write $z^2 = R^2 - r^2$ and find

$$ds^2 = \left(1 + \frac{r^2}{R^2 - r^2}\right) dr^2 + r^2 d\theta^2. \tag{5}$$

We can further write

$$1 + \frac{r^2}{R^2 - r^2} = \frac{R^2 - r^2 + r^2}{R^2 - r^2} = \frac{1}{1 - (r/R)^2}. \tag{6}$$

Therefore in the end the metric takes the form

$$ds^2 = \frac{dr^2}{1 - (r/R)^2} + r^2 d\theta^2. \tag{7}$$