A Simpler Porosity Bridging Law
Based on a Parallel Resistance Analogy

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1. Porosity Reduction of Opacity in a Scattering Medium

Spurred by attempts to model the "crowding effect" in Rich and Dan's Monte Carlo sims of porous media, I've been developing a somewhat modified model for porous transport. In contrast to the previous focus on the *absorption* properties of an *isolated* blob, this new approach is based on how the *scattering* in the medium is modified by the larger path length associated with *space between* the *ensemble* of blob structures. In this way, it is closer in spirit to the Rosseland mean opacity approach, which defines frequency averages in terms of the *inverse* opacity, which scales with the path length.

Another, even simpler analogy would be the nature of current flow through a parallel circuit, wherein the effective resistance is obtained from the inverse of the inverse sum of the individual resistors. In this analogy, the opacity can be thought of as analogous to the resistance, with its inverse, which is proportional to mean path length, analogous to the conductance.

Instead of the previous focus on the opacity per unit mass, κ, it is more convenient in this formulation to deal with the opacity per unit volume, $\kappa\rho$, which has units of an inverse length, namely the inverse of the associated mean-free-path. The new model defines the mean net path of the effective opacity by adding the mean path for the microscopic opacity, $1/\kappa\rho$, and the extra mean path, $H = 1/n\sigma$, for the assumed clumping of the medium into an ensemble of blobs with individual cross section σ and overall number density per unit volume n. This gives

$$\frac{1}{\kappa_{eff}\,\rho} = \frac{1}{\kappa\rho} + H\,. \tag{1}$$

For blobs of size ℓ and separation L, the blob cross-section scales as $\sigma = \ell^2$, while the blob volume density scales as $n = 1/L^3$. We thus see that in this case the associated blob mean-free-path is just our usual "porosity length", $H = L^3/\ell^2$.

Solving for the effective opacity (per unit mass), this now gives for the porosity-related scattering reduction in opacity,

$$\frac{\kappa_{eff}}{\kappa} = \frac{1}{1+\tau_b}\,, \tag{2}$$

where $\tau_b \equiv \kappa\rho H$ is again the optical thickness of individual blobs. Figure 1 compares this against the old, single-blob-absorption bridging law,

$$\frac{\kappa_{eff}^{abs}}{\kappa} = \frac{1-e^{-\tau_b}}{\tau_b}\,. \tag{3}$$

Note that both laws have the same asymptotic forms in the opposing limits of optically thin and optically thick blobs. Namely, for thin blobs with $\tau_b \ll 1$, they simply recover the microscopic opacity $\kappa_{eff} \rightarrow \kappa$, while for thick blobs with $\tau_b \gg 1$, they give a (mass) opacity set by the ratio of the blob cross section ℓ^2 to blob mass $m_b = \rho L^3$, i.e. $\kappa_{eff} \rightarrow 1/\rho H$. However, figure 1 shows that the new scattering form drops somewhat more sharply for moderately small $\tau_b \lesssim 1$, with Taylor expansion giving $\kappa_{eff} \sim 1-\tau_b$ instead of the $\kappa_{eff}^{abs} \sim 1-\tau_b/2$ of the old absorption form.

While both full bridging forms are quite elementary, the simple ratio character of the new form, without any exponential function, makes it even more convenient for certain analyses, for example in deriving the wind optical depth for X-ray absorption, as developed in my recent email notes to David Cohen. Actually, for that case of bound-free absorption, the basic absorption model is actually somewhat more appropriate. But in the context of electron scattering in super-Eddington atmospheres, the new scattering analysis seems closer to the physics at hand.

2. Accounting for the Crowding Effect

In fact, the focus here on the path-length *between* the blobs, instead of the absorption characteristics of individual blobs, can offer us some insight on how to account for "crowding effect" when the size of the blobs is no longer assumed to be small compared to their separation.

To see how this can come about, consider the quite artificial, idealized model of *cubic blobs* of side ℓ spaced in a perfectly regular *cubic lattice* with separations L between the centers of neighboring blobs. The cross section of the blobs is again of order $\sigma \sim \ell^2$ (ignoring subtleties of different projections). However, in considering the free path between the blobs, it now seems more appropriate to focus only on the *net empty* volume $V = L^3 - l^3$, for which the associated path length becomes

$$H'_{cubes} = V/\sigma = H(1-f')\,, \tag{4}$$

where $f' \equiv (\ell/L)^3$ is the blob volume filling ratio. We thus see that in the crowding-case that $\ell \rightarrow L$, the space between the blobs fills up, giving $f' \rightarrow 1$ and so $H' \rightarrow 0$. As expected, this means the porosity effect vanishes for $\ell = L$. This special reduction to zero porosity is thus quite "sharp" as the blobs just fit in the vanishing space between them. But this simple model cannot readily accommodate *overlapping* blobs, such as would occur for $\ell > L$.

For a somewhat more realistic model of *randomly* spaced *spherical* blobs, this overlapping limit can be expected to consist of both regions of multiple blob density from overlapping spheres, interspersed perhaps with still some empty gaps between the spheres, at least if the blob size is not too much bigger than the characteristic separation, i.e. $\ell \gtrsim L$. Still, one expects that in the limit $\ell \gg L$ the overlapping blobs should resemble more and more a smooth medium, for which the porosity length should asymptotically vanish. This can be accommodated, for example, by a simple scaling like

$$H' = \frac{H}{1+f'} = H(1-f) = \frac{\ell}{f' + f'^2} \, . \tag{5}$$

where the second equality introduces the scaling in terms of the blob filling *fraction* $f = f'/(1+f')$, and the last equality recalls the original definition of porosity length as the ratio of the blob scale to blob filling ratio, $H \equiv \ell/f'$. The resulting crowding-corrected form for the effective opacity is This now gives for the porosity-related scattering reduction in opacity,

$$\frac{\kappa_{eff}}{\kappa} = \frac{1}{1 + \tau_b/(1+f')} \, . \tag{6}$$

Figure plots this vs. blob optical thickness τ_b for various filling ratios ranging from $f' = 0$ to $f' = 16$. Note that the associated scale ratio are given by $\ell/L = f'^{1/3}$.

3. Concluding Remarks

This scaling is still rather ad hoc, and it will be helpful to compare it to results from Monte Carlo simulations. I suspect these might show a stronger saturation of the porosity effect as crowding increases, and so I'm still thinking of what scalings could ensure a rapid approach to the uniform limit $\kappa_{eff} = \kappa$ for $f' \gg 1$, independent of how big τ_b gets. But for now, I think this approach at least gives us a useful way to think about porous scattering in both the crowded and uncrowded limit.

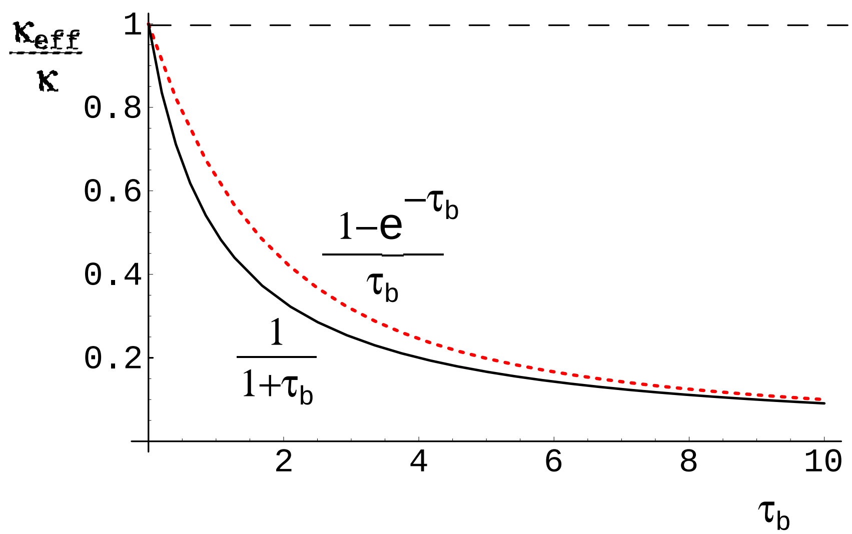

Fig. 1.— Absorption (red-dotted curve; eqn. (3) and scattering (black solid curve; eqn. (2)) bridging laws for effective opacity reduction from porosity, κ_{eff}/κ, plotted vs. blob optical thickness, τ_b.

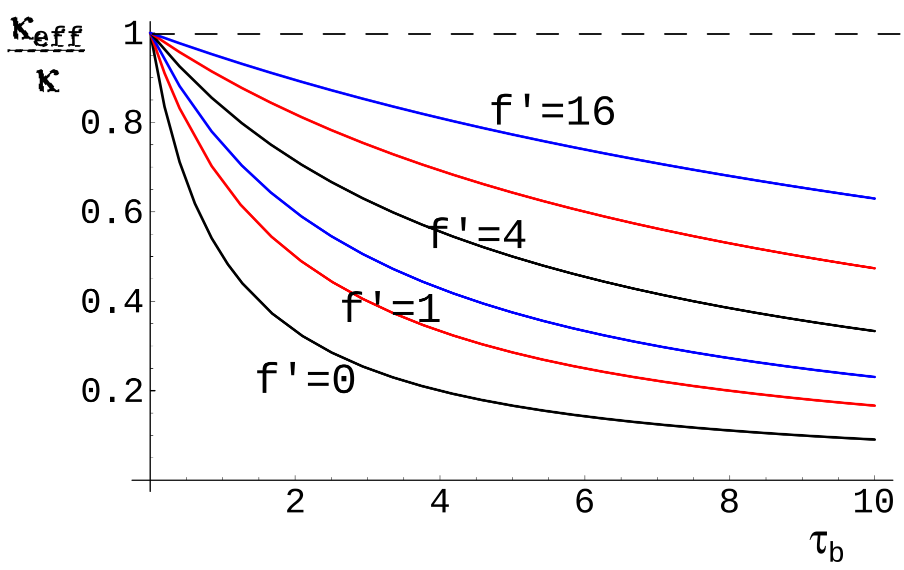

Fig. 2.— Crowding corrected effec tive opacity, κ_{eff}/κ, vs. blob optical thickness, τ_b, for various values of the volume filling ratio, f' =0, 1, 2, 4, 8, and 16, increasing from bottom to top. Note that the associated scale ratios are given by $\ell/L = f'^{1/3}$.